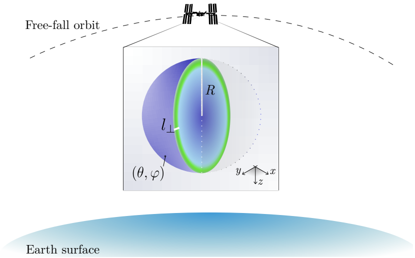
Quantum statistics and BKT transition of a shell-shaped superfluid

Andrea Tononi

Laboratoire de Physique Théorique et Modèles Statistiques (Orsay)

Dipartimento di Fisica e Astronomia "Galileo Galilei", Università di Padova;

Young Seminars SIFS



In collaboration with
A. Pelster, F. Cinti, L. Salasnich.

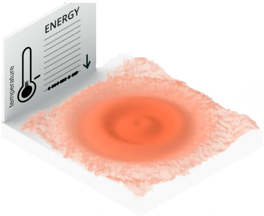
This presentation will be on
www.andreatononi.com

Outline

- Introduction and motivation
- Bose-Einstein condensation in shell-shaped condensates
  - Spherical case
  - Ellipsoidal case
- Superfluidity in 2D: BKT transition
  - Spherical superfluid
- Conclusions

Quantum gases

Bose & Einstein [1924-25]:
predicted the phenomenon of
Bose-Einstein condensation



macroscopic occupation of the same lowest-energy **single-particle state** of a many-body system of identical bosons

(occupation numbers controlled by T, and by n)

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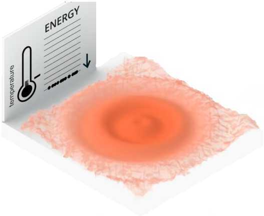
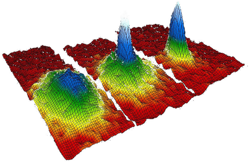


macroscopic occupation of the same lowest-energy **single-particle state** of a many-body system of identical bosons

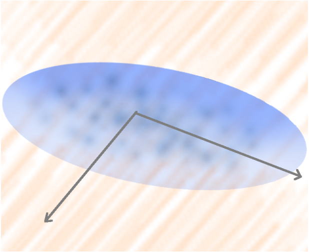
(occupation numbers controlled by T, and by n)

Cornell & Wieman, Ketterle, [1995]:
observation of BEC by trapping and
cooling alkali-metal atoms

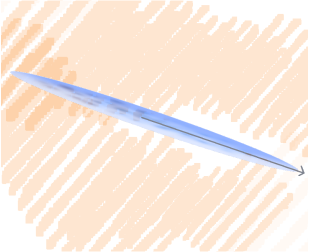


Low-dimensional quantum gases

(2D)



(1D)



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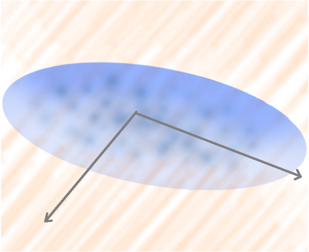

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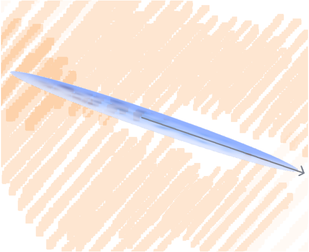

Quantum many-body physics has been studied **consistently** only in *"flat"* low-dimensional configurations

What about *curved* geometries?

Shell-shaped quantum gases (rf-induced adiabatic potentials)

Theoretical proposal of [Zobay, Garraway, PRL **86**, 1195 (2001)]**:**

confine the atoms with $B_0(\vec{r})$, and $B_{rf}(\vec{r}, t)$, yielding

$$U(\vec{r}) = M_F \sqrt{\left[\sum_i \frac{m}{2}\,{\omega_i}^2 x_i^2 - \hbar\Delta\right]^2 + (\hbar\Omega)^2}$$

ω_i: frequencies of the bare harmonic trap

Δ: detuning from the resonant frequency

Ω: Rabi frequency between coupled levels

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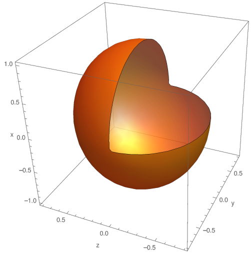
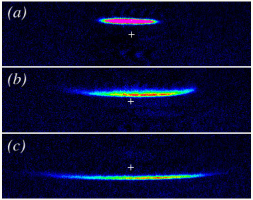
Ω: Rabi frequency between coupled levels

Minimum of $U(\vec{r})$ for
$${\omega_x}^2 x^2 + {\omega_y}^2 y^2 + {\omega_z}^2 z^2 = \frac{2\hbar\Delta}{m}.$$



Shell-shaped quantum gases

On Earth...



[Colombe *et al.*, EPL **67**, 593 (2004)]

Shell-shaped quantum gases

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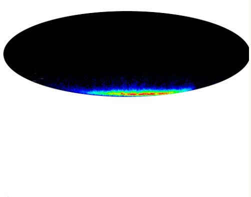

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Shell-shaped quantum gases, in microgravity

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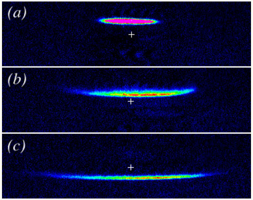
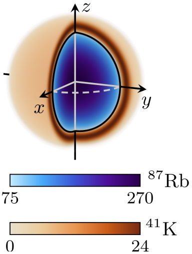


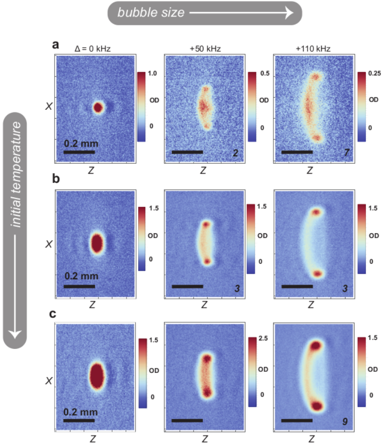
[Colombe *et al.*, EPL **67**, 593 (2004)]

...and other proposals:



[Wolf *et al.*, arXiv:2110.15247]

...in microgravity:



[Carollo *et al.*, arXiv:2108.05880]

(experiments on CAL, on ISS)

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Bose-Einstein condensation on the surface of a sphere

Noninteracting case, single particle on a sphere of radius R:

$$\frac{\hat{L}^2}{2mR^2}\,\mathcal{Y}_{l,m_l}(\theta,\varphi) = \epsilon_l\,\mathcal{Y}_{l,m_l}(\theta,\varphi),$$

with $\epsilon_l = \frac{\hbar^2}{2mR^2}\,l(l+1)$ and $m_l = -l, \dots, +l$.

Particle number: $N = N_0 + \sum_{l=1}^{+\infty} \frac{(2l+1)}{e^{(\epsilon_l-\epsilon_0)/(k_B T)}-1}$

[AT, Salasnich, PRL **123**, 160403 (2019)]

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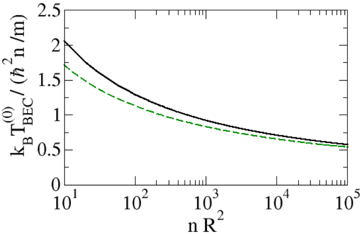
$$\frac{k_B T_{\mathrm{BEC}}}{\hbar^2 n/m} = \frac{-2\pi}{\ln\left[1 - e^{-\frac{\hbar^2/(mR^2)}{k_B T_{\mathrm{BEC}}}}\right]}$$



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Uniform interacting bosons on the surface of the sphere

$$\Omega = -\frac{1}{\beta}\ln(\mathcal{Z}), \qquad \mathcal{Z} = \int \mathcal{D}[\bar{\psi},\psi]\; e^{-\frac{S[\bar{\psi},\psi]}{\hbar}},$$

with $S[\bar{\psi},\psi] = \int_0^{\beta\hbar} d\tau \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, R^2 \sin\theta\, \mathcal{L}(\bar{\psi},\psi)$, and

$$\mathcal{L} = \bar{\psi}(\theta,\varphi,\tau)\left(\hbar\partial_\tau + \frac{\hat{L}^2}{2mR^2} - \mu\right)\psi(\theta,\varphi,\tau) + \frac{g_0}{2}|\psi(\theta,\varphi,\tau)|^4.$$

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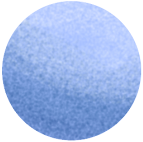
Bogoliubov theory:

$$\psi(\theta,\varphi,\tau) = \psi_0 + \eta(\theta,\varphi,\tau)$$

$\eta(\theta,\varphi,\tau)$



Bogoliubov spectrum

$$E_l^{\mathrm{B}} = \sqrt{\epsilon_l(\epsilon_l + 2\mu)}$$

ψ_0

[AT, Salasnich, PRL **123**, 160403 (2019)]

Equation of state of a spherical Bose gas

Implementing the Bogoliubov theory, we calculate

- Ω
- n_0/n
- T_{BEC}
- $n = \partial_\mu[\Omega/(4\pi R^2)]$

*: [AT, Pelster, Salasnich, arXiv:2104.04585], [AT, submitted to PRA]

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In particular, we regularize the grand potential Ω through the analysis of scattering theory*. The number density reads:

$$n = \frac{m\mu}{4\pi\hbar^2} \ln\left\{ \frac{4\hbar^2[1-\alpha(\mu)]}{m\mu\, a_s^2\, e^{2\gamma+1+\alpha(\mu)}} \right\} + \frac{1}{4\pi R^2} \sum_{l=1}^{\infty} \sum_{m_l=-l}^{l} \frac{\epsilon_l}{E_l^B} \frac{1}{e^{\beta E_l^B}-1},$$

where we introduce the positive function $\alpha(\mu) = 1 - \frac{\mu}{\mu + E_1^B + \epsilon_1}$.

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Complete description of the quantum statistical properties of a weakly-interacting spherical Bose gas

*: [AT, Pelster, Salasnich, arXiv:2104.04585], [AT, submitted to PRA]

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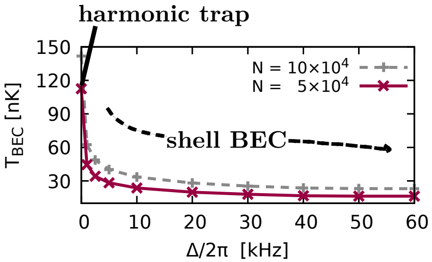
Bose-Einstein condensation in ellipsoidal bubbles

In [AT, Cinti, Salasnich, PRL **125**, 010402 (2020)], we modeled the microgravity experiments ([arXiv:2108.05880])

For the realistic trap parameters:

T_{BEC} drops quickly with $\Delta \propto$ shell area



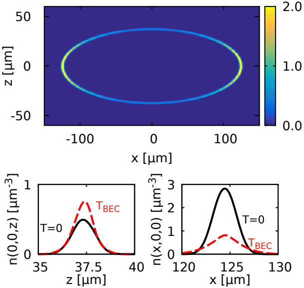
$N \sim 10^5$, $T_{\mathrm{BEC}} \sim 30\,\mathrm{nK}$

Difficult to reach fully-condensate regime...

⇒ Finite-temperature properties are highly relevant

Density distribution

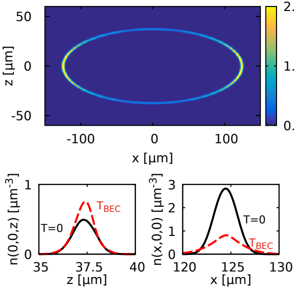
Condensate vs thermal density



[AT, Cinti, Salasnich, PRL **125**, 010402 (2020)]

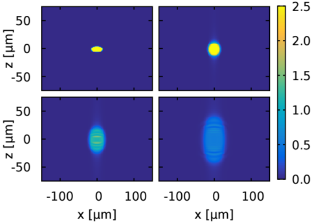
Density distribution and free expansion

Condensate vs thermal density

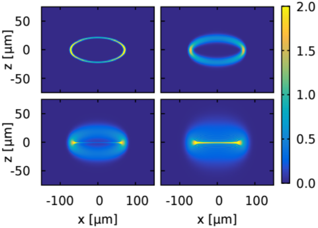


[AT, Cinti, Salasnich, PRL **125**, 010402 (2020)]

Harmonic trap



Bubble trap



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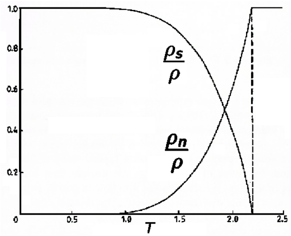
Superfluidity: flow without friction

Landau two-fluid model:

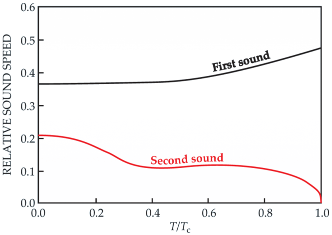


[Landau J. Phys. (USSR) **5**, 71 (1941)]

predicts two sound modes

$$c_{1,2} = \left[\frac{v_A^2 + v_L^2}{2} \pm \sqrt{\left(\frac{v_A^2 + v_L^2}{2}\right)^2 - v_L^2 v_T^2} \right]^{1/2},$$

$$v_{\{A,T\}} = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_{\{\tilde{s},T\}}}, \quad v_L = \sqrt{\frac{\rho_s T \tilde{s}^2}{\rho_n \tilde{c}_V}}$$



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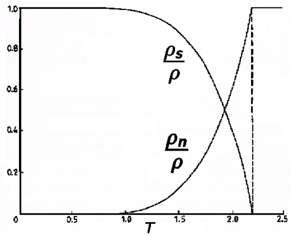

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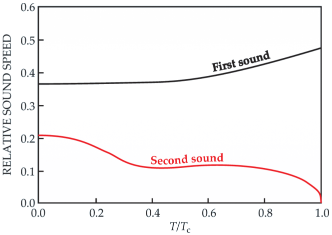
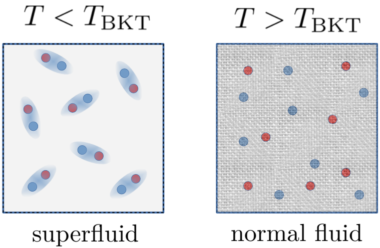


superfluid transition in 2D?

Berezinskii-Kosterlitz-Thouless transition (BKT)



superfluid transition in 2D

[Nelson, Kosterlitz, PRL **39**, 1201 (1977)]

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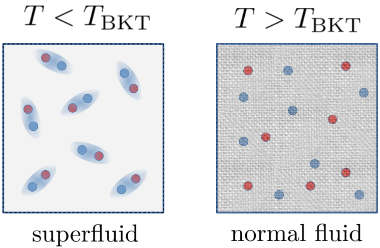


superfluid transition in 2D

[Nelson, Kosterlitz, PRL **39**, 1201 (1977)]

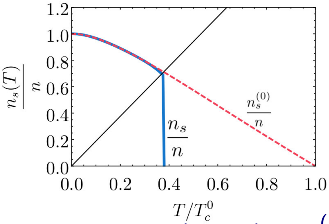
RG equations of a flat superfluid

$$\frac{dK^{-1}(\ell)}{d\ell} = -4\pi^3 y^2(\ell)$$

$$\frac{dy(\ell)}{d\ell} = [2 - \pi K(\ell)]\, y(\ell)$$

Adimensional parameters

$K(\ell) = \frac{\hbar^2 n_s(\ell)}{m k_B T}$; $y(\ell) \sim e^{-\beta \mu_v(\ell)}$;



$\rightarrow$ From bare $n_s(\ell = 0) = n_s^{(0)}$ to renormalized $n_s = n_s(\ell = \infty)$

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BKT transition in spherical superfluids

RG equations of a spherical superfluid

$$\frac{dK^{-1}(\theta)}{d\ell(\theta)} = -4\pi^3 y^2(\theta)$$

$$\frac{dy(\theta)}{d\ell(\theta)} = [2 - \pi K(\theta)]\, y(\theta)$$

RG scale?

$$\ell(\theta) = \ln[2R\sin(\theta/2)/\xi]$$

Distance between vortices:

$2R\sin(\theta/2) \in [\xi, 2R]$

...in 3D space

[AT, Pelster, Salasnich, arXiv:2104.04585]

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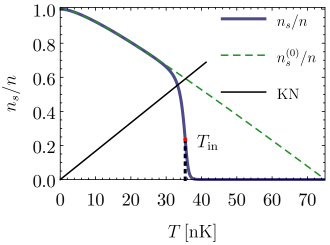
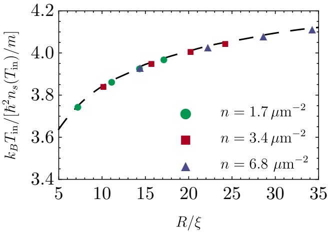
Finite system size ⇒ **smooth** vanishing of n_s



for different n, shell widths $T_{\rm in}$ scales with R/ξ



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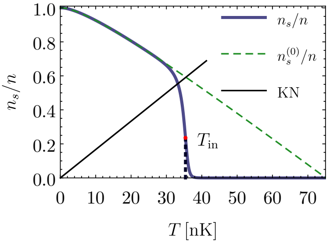
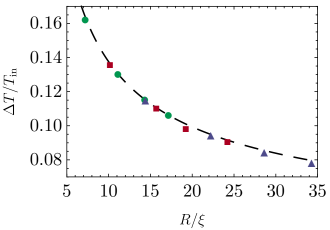


for different n, shell widths $\Delta T/T_{\text{in}} \propto \ln^{-2}(R/\xi)$



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Hydrodynamic modes in spherical superfluids

Given the equation of state **and** the superfluid density, we extend the Landau two-fluid model to the spherical case.

[AT, Pelster, Salasnich, arXiv:2104.04585]

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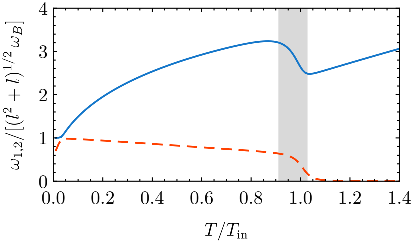
Given the equation of state **and** the superfluid density, we extend the Landau two-fluid model to the spherical case.

Frequencies of the hydrodynamic modes:

$$\omega_{1,2}^2 = \left[\frac{l(l+1)}{R^2}\right]\left[\frac{v_A^2+v_L^2}{2} \pm \sqrt{\left(\frac{v_A^2+v_L^2}{2}\right)^2 - v_L^2 v_T^2}\right]$$

ω_1, ω_2 are the main quantitative probe of BKT physics



$$v_{\{A,T\}} = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_{\{\tilde{s},T\}}}, \quad v_L = \sqrt{\frac{\rho_s T \tilde{s}^2}{\rho_n \tilde{c}_V}}$$

[AT, Pelster, Salasnich, arXiv:2104.04585]

Outline

- Introduction and motivation
- Bose-Einstein condensation in shell-shaped condensates
  - Spherical case
  - Ellipsoidal case
- Superfluidity in 2D: BKT transition
  - Spherical superfluid
- Conclusions

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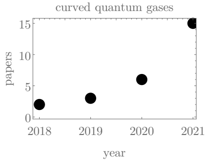
– Curvature in quantum gases (and in cond. mat.): a new research direction.

The scientific community has just started exploring shell-shaped BECs



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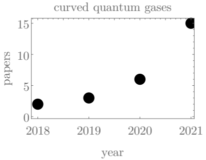

– in spherical condensates: curvature ≈ finite-size, but consequences on superfluidity

– interesting perspectives with ellipsoidal shells

Thank you for your attention!

References

- A. Tononi, F. Cinti, and L. Salasnich, *Quantum Bubbles in Microgravity*, Physical Review Letters **125**, 010402 (2020).
- A. Tononi and L. Salasnich, *Bose-Einstein Condensation on the Surface of a Sphere*, Physical Review Letters **123**, 160403 (2019).
- A. Tononi, A. Pelster, and L. Salasnich, arXiv:2104.04585