Scattering theory and equation of state of a spherical 2D Bose gas

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Palaiseau-Florence Workshop
on Ultracold Atoms

21/04/2022

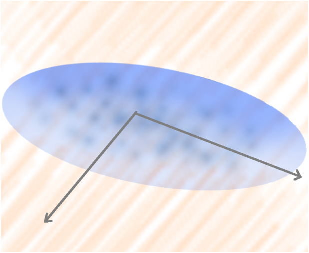
based on
[A. Tononi, PRA **105**, 023324 (2022)]

Outline

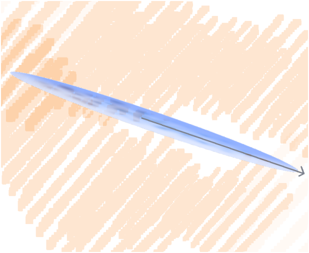
- Introduction
- Equation of state of a 2D spherical Bose gas
- Derivation of the equation through scattering theory
- Application: hydrodynamic modes
- Conclusions

Low-dimensional quantum gases

(2D)



(1D)



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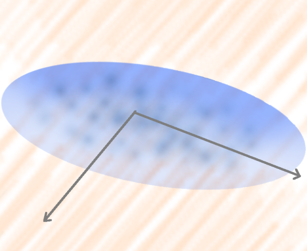

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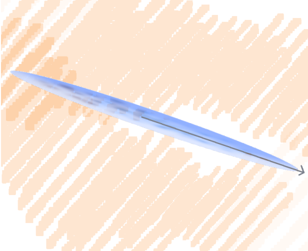

Quantum gases and their many-body properties have been studied **consistently** only in *"flat"* low-dimensional configurations

What about *curved* geometries?

Bubble trap (rf-induced adiabatic potentials)

Theoretical proposal of [Zobay, Garraway, PRL **86**, 1195 (2001)]**:**
confine the atoms with $B_0(\vec{r})$, and $B_{rf}(\vec{r}, t)$, yielding

$$U(\vec{r}) = M_F \sqrt{\left[\sum_i \frac{m}{2}\,{\omega_i}^2 x_i^2 - \hbar\Delta\right]^2 + (\hbar\Omega)^2}$$

ω_i: frequencies of the bare harmonic trap

Δ: detuning from the resonant frequency

Ω: Rabi frequency between coupled levels

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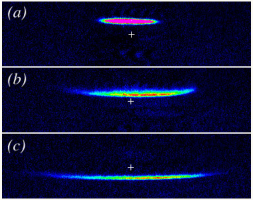
Ω: Rabi frequency between coupled levels

Minimum of $U(\vec{r})$ for
$${\omega_x}^2 x^2 + {\omega_y}^2 y^2 + {\omega_z}^2 z^2 = \frac{2\hbar\Delta}{m}.$$

Quantum bubbles

On Earth...



[Colombe *et al.*, EPL **67**, 593 (2004)]

Quantum bubbles

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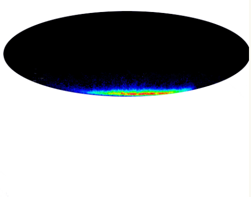

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Quantum bubbles, in microgravity

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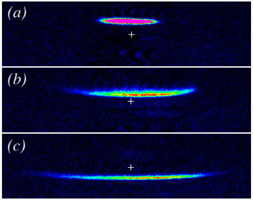

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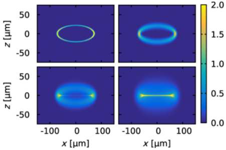
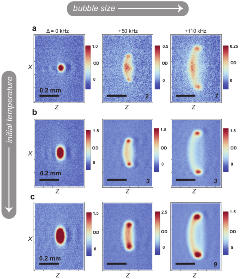
PHYSICAL REVIEW LETTERS **125**, 010402 (2020)

Quantum Bubbles in Microgravity

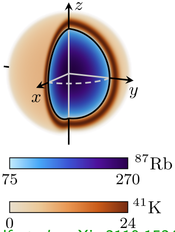
A. Tononi[1,*] F. Cinti[2,3,4,†] and L. Salasnich[1,5,‡]



...in microgravity:



[Carollo *et al.*, arXiv:2108.05880]



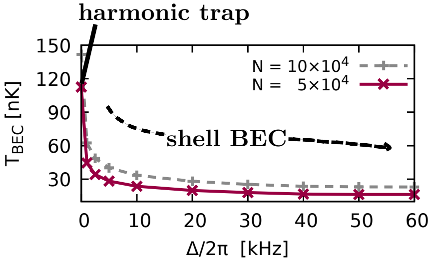
[Wolf *et al.*, arXiv:2110.15247]

Bose-Einstein condensation in ellipsoidal bubbles

Modeling of microgravity experiments in
[AT, Cinti, Salasnich, PRL **125**, 010402 (2020)]

For the realistic trap parameters ([Lundblad *et al.*, npj Microgravity **5**, 30 (2019)]):

T_{BEC} drops quickly with $\Delta \propto$ shell area



$N \sim 10^5, \; T_{\mathrm{BEC}} \sim 30\,\mathrm{nK}$

Interplay of T and $T^{(0)}_{\mathrm{BEC}}$: [Rhyno, et al. PRA **104**, 063310 (2021)]

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Equation of state of a spherical Bose gas

Implementing the Bogoliubov theory, we calculated T_{BEC}, n_0/n, Ω.

[AT, Salasnich, BEC on the surface of a sphere, PRL **123**, 160403 (2019)]

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Recently, through the analysis of scattering theory*...

equation of state:

$$n = \frac{m\mu}{4\pi\hbar^2} \ln\left\{ \frac{4\hbar^2[1-\alpha(\mu)]}{m\mu\, a_s^2\, e^{2\gamma+1+\alpha(\mu)}} \right\},$$

with $\alpha(\mu) = 1 - \frac{\mu}{\mu + E_1^B + \epsilon_1}$,

$E_l^{\mathrm{B}} = \sqrt{\epsilon_l(\epsilon_l + 2\mu)}$,

$\epsilon_l = \hbar^2 l(l+1)/(2mR^2)$

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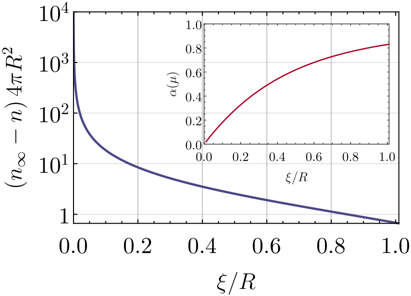

$$\left(n \equiv n_\infty = \frac{m\mu}{4\pi\hbar^2} \ln\left(\frac{4\hbar^2}{m\mu\, a_s^2\, e^{2\gamma+1}}\right) \text{ at } R=\infty,\ \alpha = 0 \right)$$

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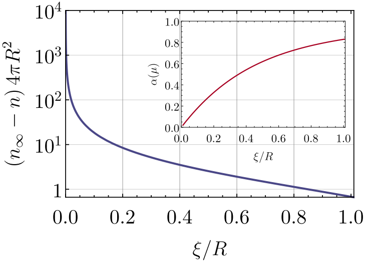

Comments:

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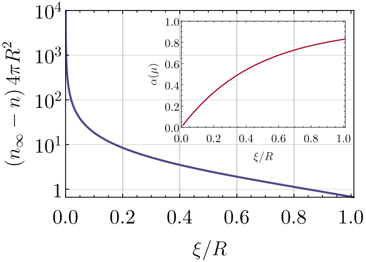

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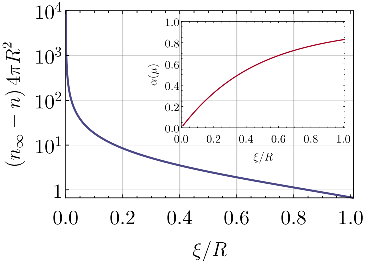

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- extandable (in principle) to other geometries

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Let us see how the equation of state is derived

Bogoliubov theory of a spherical gas

Uniform bosons on the surface of the sphere

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}, \psi] \; e^{-\frac{S[\bar{\psi},\psi]}{\hbar}}, \qquad \Omega = -\frac{1}{\beta} \ln(\mathcal{Z})$$

where

$$S[\bar{\psi}, \psi] = \int_0^{\beta\hbar} d\tau \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, R^2 \sin\theta \, \mathcal{L}(\bar{\psi}, \psi)$$

is the Euclidean action, and

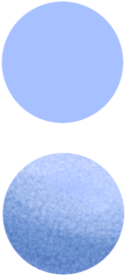
$$\mathcal{L} = \bar{\psi}(\theta, \varphi, \tau)\left(\hbar\partial_\tau + \frac{\hat{L}^2}{2mR^2} - \mu\right)\psi(\theta, \varphi, \tau) + \frac{g_0}{2}|\psi(\theta, \varphi, \tau)|^4$$

is the Euclidean Lagrangian.

Bogoliubov theory of a spherical gas

ψ_0

$\eta(\theta, \varphi, \tau)$



Bogoliubov theory:

$$\psi(\theta, \varphi, \tau) = \psi_0 + \eta(\theta, \varphi, \tau)$$

Performing the Gaussian integral on $\sim \eta^2$ terms, we get

$$\Omega = -\,(4\pi R^2)\,\frac{\mu^2}{2g_0} + \frac{1}{2}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l}(E_l^{\mathrm{B}} - \epsilon_l - \mu),$$

with $E_l^{\mathrm{B}} = \sqrt{\epsilon_l(\epsilon_l + 2\mu)}$, and $\epsilon_l = \hbar^2 l(l+1)/(2mR^2)$.

[AT, Salasnich, PRL **123**, 160403 (2019)]

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Problem: the zero-point energy diverges logarithmically at large l:

$$\frac{1}{2}\int_1^{l_c} \mathrm{d}l\,(2l+1)\left(E_l^{\mathrm{B}} - \epsilon_l - \mu\right) \sim \ln(l_c)$$

Solution: g_0 scales with l_c!

To see this, we need to discuss scattering theory

Scattering theory on the sphere

For a particle with reduced mass on the sphere, the interacting scattering problem reads [Zhang, Ho, J. Phys. B **51**, 115301 (2018)]

$$\hat{H}_0 \Psi_\nu^\mu(\theta, \varphi) = \mathcal{E}_\nu \Psi_\nu^\mu(\theta, \varphi), \qquad \textbf{when} \quad \theta > r_0/R$$

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We identify (**it is a shortcut**, see [AT, PRA **105**, 023324 (2022)] for all steps)

$$g_0 \approx f_0(\mathcal{E}_{l_c}) \approx -\frac{2\pi\hbar^2}{m} \frac{1}{\ln\left[l_c\, a_s e^{\gamma_E}/(2R) \right]}$$

Regularized equation of state

Putting $g_0 = -\frac{2\pi\hbar^2}{m}\frac{1}{\ln\left[l_c\, a_s e^{\gamma_E}/(2R)\right]}$ into

$$\Omega = -\,(4\pi R^2)\,\frac{\mu^2}{2g_0} + \frac{1}{2}\int_1^{l_c} \mathrm{d}l\,(2l+1)\,(E_l^{\mathrm{B}} - \epsilon_l - \mu),$$

the $\ln(l_c)$ divergence disappears, and we obtain the equation of state:

$$n = -\frac{1}{4\pi R^2}\frac{\partial\Omega}{\partial\mu} = \frac{m\mu}{4\pi\hbar^2}\,\ln\left\{\frac{4\hbar^2[1-\alpha(\mu)]}{m\mu\, a_s^2\, e^{2\gamma+1+\alpha(\mu)}}\right\}$$

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Knowing the equation of state **and** the superfluid density, we extend the Landau two-fluid model to the spherical case.

[AT, Pelster, Salasnich, PRR **4**, 013122 (2022)]

Application: hydrodynamic modes

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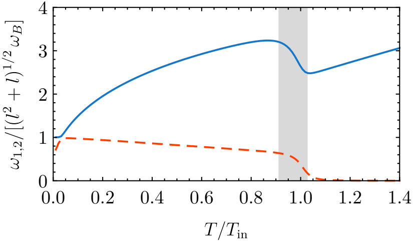
Frequencies of the hydrodynamic modes:

$$\omega_{1,2}^2 = \left[\frac{l(l+1)}{R^2}\right]\left[\frac{v_A^2 + v_L^2}{2} \pm \sqrt{\left(\frac{v_A^2 + v_L^2}{2}\right)^2 - v_L^2 v_T^2}\right]$$

ω_1, ω_2 are the main quantitative probe of superfluid BKT transition

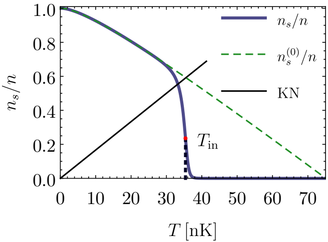


$$v_{\{A,T\}} = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_{\{\tilde{s},T\}}}, \quad v_L = \sqrt{\frac{\rho_s T \tilde{s}^2}{\rho_n \tilde{c}_V}}$$

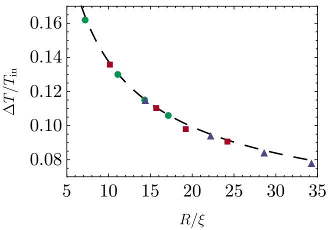
[AT, Pelster, Salasnich, PRR **4**, 013122 (2022)]

Superfluid BKT transition in a spherical superfluid

Finite system size ⇒
smooth vanishing of n_s



$\Delta T/T_{\rm in} \propto \ln^{-2}(R/\xi)$



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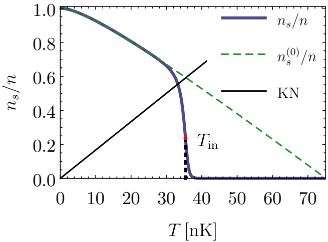

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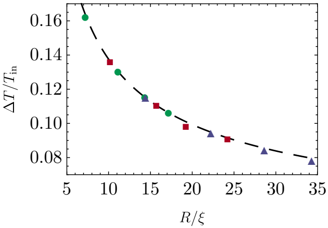

Renormalization group equations

$$\frac{dK^{-1}(\theta)}{d\ell(\theta)} = -4\pi^3 y^2(\theta)$$

$$\frac{dy(\theta)}{d\ell(\theta)} = [2 - \pi K(\theta)]\, y(\theta)$$

RG scale: $\ell(\theta) = \ln[2R\sin(\theta/2)/\xi]$

describe how the superfluid density ($\propto K$) is renormalized by the thermally excited vortices with chemical potential $\sim -\ln(y)$

$$E^{(\text{vor})} = \sum_{i=1}^{M} q_i^2\, \mu_v - K^{(0)} \sum_{i\neq j=1}^{M} q_i q_j \ln\left[2R\sin(\gamma_{ij}/2)\xi\right]$$

[AT, Pelster, Salasnich, PRR **4**, 013122 (2022)]

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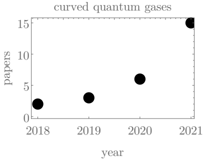
– Curvature in quantum gases (and in cond. mat.): a new research direction.

The scientific community has just started exploring shell-shaped BECs, both **experimentally** and **theoretically**



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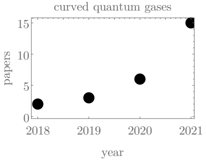

  - in spherical condensates: curvature $\approx$ finite-size for BEC, but consequences on superfluidity

  - interesting perspectives with ellipsoidal shells

Thank you for your attention!

References:

— AT, Salasnich, PRL **123**, 160403 (2019)

— AT, Cinti, Salasnich, PRL **125**, 010402 (2020)

— AT, Pelster, Salasnich, PRR **4**, 013122 (2022)

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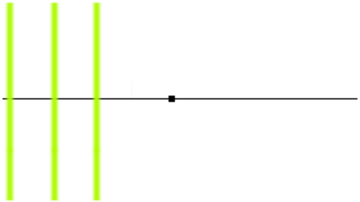
Backup slides

Scattering theory

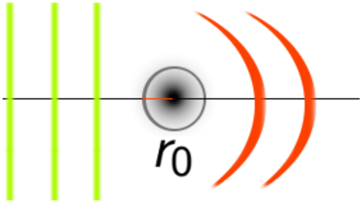
Noninteracting scattering problem: $\hat{H}_0 |\phi\rangle = \mathcal{E}_0 |\phi\rangle$



and we suppose that $|\phi\rangle$, and $\mathcal{E}_0$ are known

Scattering theory

Interacting scattering problem: $(\hat{H}_0 + \hat{V}) |\Psi\rangle = \mathcal{E}_0 |\Psi\rangle$



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and imposing $\Psi_\nu^0(a_s/R,\varphi) = 0$:

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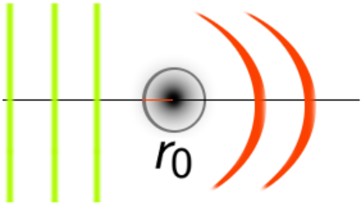
One could set $f_0 \approx g_0$, but how we fix ν?

Let us reconsider the scattering problem and find a condition to determine $g_0(l_c, a_s)$.

Scattering theory

Interacting scattering problem: $(\hat{H}_0 + \hat{V})\,|\Psi\rangle = \mathcal{E}\,|\Psi\rangle$



whose solution $\left|\Psi^{(+)}\right\rangle$ is given by the Lippmann-Schwinger equation

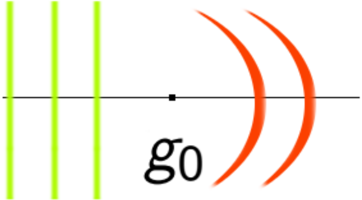
$$\mathcal{T} = \hat{V} + \hat{V}\,\frac{1}{\mathcal{E}_0 - \hat{H}_0 + i\eta}\,\hat{\mathcal{T}},$$

where $\hat{\mathcal{T}}\,|\phi\rangle = \hat{V}\,\left|\Psi^{(+)}\right\rangle$.

[Lippmann, Schwinger, PR **79**, 469 (1950)]

Scattering problem on the sphere

We consider the interatomic potential $\hat{V}_0 = \tilde{g}_0\, \delta(1 - \cos\theta)\, \delta(\varphi)$:



and calculate $\mathcal{T}_{l',l_0} = \langle l', m_l' = 0 |\, \hat{\mathcal{T}}\, | l_0, m_{l_0} = 0 \rangle$ (s-wave scattering).

We get the Born series

$$\mathcal{T}_{l',l_0} = \tilde{g}_0 \frac{\sqrt{(2l'+1)(2l_0+1)}}{4\pi} \left[1 + \sum_{l=0}^{\infty} \frac{\sqrt{2l+1}}{\sqrt{2l_0+1}} \frac{\mathcal{T}_{l,l_0}}{\mathcal{E}_{l_0} - \mathcal{E}_l + i\eta} \right],$$

Scattering problem on the sphere

Summing the Born series, we get the renormalized interaction strength

$$\frac{\sqrt{(2l'+1)(2l_0+1)}}{4\pi\mathcal{T}_{l',l_0}} = \frac{1}{\tilde{g}_e(\mathcal{E}_{l_0}+i\eta)} = \frac{1}{\tilde{g}_0} + \frac{1}{4\pi}\sum_{l=0}^{l_c}\frac{2l+1}{\mathcal{E}_l-\mathcal{E}_{l_0}-i\eta},$$

By calculating the sum as integral and setting $\tilde{g}_e(\mathcal{E}_{l_0}) = f_0(\mathcal{E}_\nu)$, we get

$$g_0 = -\frac{2\pi\hbar^2}{m}\frac{1}{\ln\left[\sqrt{l_c(l_c+1)}\,a_s e^{\gamma_E}/(2R)\right]}$$

Regularized equation of state

Putting

$$g_0 = -\frac{2\pi\hbar^2}{m}\frac{1}{\ln\left[\sqrt{l_c(l_c+1)}\,a_s e^{\gamma_E}/(2R)\right]}$$

into

$$\Omega(T=0) = -\,(4\pi R^2)\,\frac{\mu^2}{2g_0} + \frac{1}{2}\int_1^{l_c} \mathrm{d}l\,(2l+1)\,(E_l^{\mathrm{B}} - \epsilon_l - \mu)$$

we get the regularized equation of state

$$\frac{\Omega(T=0)}{4\pi R^2} = -\,\frac{m\mu^2}{8\pi\hbar^2}\left\{\ln\left[\frac{4\hbar^2}{m(E_1^B+\epsilon_1+\mu)a_s^2\,e^{2\gamma+1}}\right]+\frac{1}{2}\right\} + \frac{mE_1^B}{8\pi\hbar^2}(E_1^B-\epsilon_1-\mu),$$

Regularized equation of state

Number density $n = -\frac{1}{4\pi R^2}\frac{\partial \Omega}{\partial \mu}$, yields

$$n = \frac{m\mu}{4\pi\hbar^2}\,\ln\left\{\frac{4\hbar^2[1-\alpha(\mu)]}{m\mu\, a_s^2\, e^{2\gamma+1+\alpha(\mu)}}\right\} + \frac{1}{4\pi R^2}\sum_{l=1}^{\infty}\sum_{m_l=-l}^{l}\frac{\epsilon_l}{E_l^B}\frac{1}{e^{\beta E_l^B}-1},$$

where we introduce the positive function

$$\alpha(\mu) = 1 - \frac{\mu}{\mu + E_1^B + \epsilon_1}$$

Regularized equation of state

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Here $\epsilon_1 = \hbar^2/(mR^2)$, and $E_1^B = \sqrt{\epsilon_1(\epsilon_1 + 2\mu)}$.

For $R \to \infty$: $\alpha(\mu) \to 0$, reproducing [Mora, Castin, PRA **67**, 053615 (2003)]

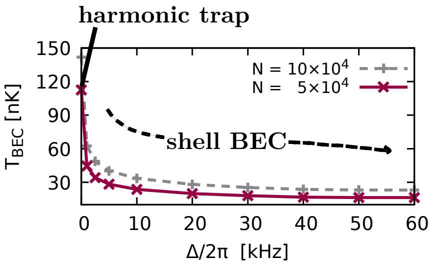
For finite R: equation of state of a finite-size curved Bose gas.

Bose-Einstein condensation in ellipsoidal bubbles

In [AT, Cinti, Salasnich, PRL **125**, 010402 (2020)], we modeled the microgravity experiments ([arXiv:2108.05880])

For the realistic trap parameters ([Lundblad *et al.*, npj Microgravity **5**, 30 (2019)]):

T_{BEC} drops quickly with $\Delta \propto$ shell area



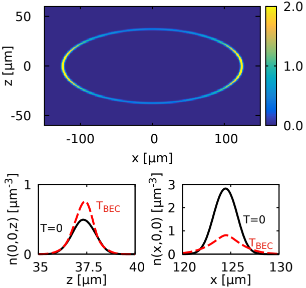
$N \sim 10^5, \; T_{\text{BEC}} \sim 30\,\text{nK}$

Difficult to reach fully-condensate regime...

$\Rightarrow$ Finite-temperature properties are highly relevant

Density distribution

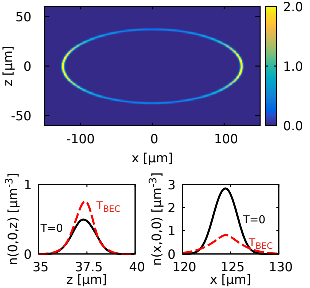
Condensate vs thermal density



[AT, Cinti, Salasnich, PRL **125**, 010402 (2020)]

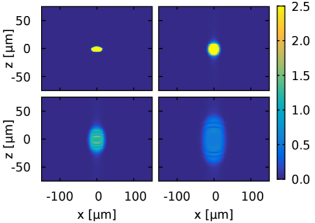
Density distribution and free expansion

Condensate vs thermal density

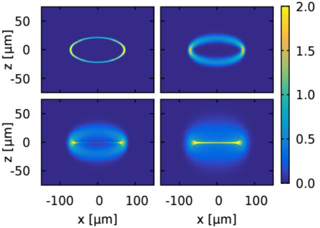


[AT, Cinti, Salasnich, PRL **125**, 010402 (2020)]

Harmonic trap



Bubble trap



Landau two-fluid model

Phenomenological description of a quantum liquid as composed by

- Superfluid: zero viscosity, no entropy
- Normal fluid: viscous, carries all the system entropy

[Landau J. Phys. (USSR) **5**, 71 (1941)]

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Total mass density:

$$\rho = \rho_s + \rho_n$$

Mass current:

$$\mathbf{j} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$$

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Hydrodynamic equations (linearized):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

$$\frac{\partial \rho \tilde{s}}{\partial t} + \rho \tilde{s} \, \nabla \cdot \mathbf{v}_n = 0$$

$$\frac{\partial \mathbf{j}}{\partial t} + \nabla P = 0$$

$$m \frac{\partial \mathbf{v}_s}{\partial t} + \nabla \mu = 0$$

[Landau J. Phys. (USSR) **5**, 71 (1941)]

Hydrodynamic modes (sound waves)

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} &= 0 \\
\frac{\partial \rho \tilde{s}}{\partial t} + \rho \tilde{s}\, \nabla \cdot \mathbf{v}_n &= 0 \\
\frac{\partial \mathbf{j}}{\partial t} + \nabla P &= 0 \\
m \frac{\partial \mathbf{v}_s}{\partial t} + \nabla \mu &= 0
\end{aligned}
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$$m \frac{\partial \mathbf{v}_s}{\partial t} + \nabla \mu = 0$$

two coupled sound equations

(III $\rightarrow \partial_t$I):

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 P$$

(I $\rightarrow$ III$, \rho, \ldots$):

$$\frac{\partial^2 \tilde{s}}{\partial t^2} = \tilde{s}^2 \frac{\rho_s}{\rho_n} \nabla^2 T$$

[Landau J. Phys. (USSR) **5**, 71 (1941)]

Hydrodynamic modes (sound waves)

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(I $\rightarrow$ III, $\rho, \ldots$):

$$\frac{\partial^2 \tilde{s}}{\partial t^2} = \tilde{s}^2 \frac{\rho_s}{\rho_n} \nabla^2 T$$

Fluctuations around the equilibrium configuration:

$$\rho \sim \rho_0 + \left(\tfrac{\partial \rho}{\partial P}\right)_T \delta P(\omega)\, e^{i\omega(t - x/c)} + \left(\tfrac{\partial \rho}{\partial T}\right)_P \delta T(\omega)\, e^{i\omega(t - x/c)},$$

$$\tilde{s} \sim \tilde{s}_0 + \left(\tfrac{\partial \tilde{s}}{\partial P}\right)_T \delta P(\omega)\, e^{i\omega(t - x/c)} + \left(\tfrac{\partial \tilde{s}}{\partial T}\right)_P \delta T(\omega)\, e^{i\omega(t - x/c)}$$

[Landau J. Phys. (USSR) **5**, 71 (1941)]

Hydrodynamic modes (sound waves)

$$\begin{cases} \delta P(\omega)\big[-c^2\big(\frac{\partial\rho}{\partial P}\big)_T + 1\big] + \delta T(\omega)\big[-c^2\big(\frac{\partial\rho}{\partial T}\big)_P\big] = 0, \\ \delta P(\omega)\big[-c^2\big(\frac{\partial\tilde{s}}{\partial P}\big)_T\big] + \delta T(\omega)\big[-c^2\big(\frac{\partial\tilde{s}}{\partial T}\big)_P + \tilde{s}^2\frac{\rho_s}{\rho_n}\big] = 0, \end{cases}$$

and setting $\det = 0$ we get the biquadratic equation:

$$c^4 - c^2\left[\left(\frac{\partial P}{\partial\rho}\right)_{\tilde{s}} + \frac{T\tilde{s}^2\rho_s}{\tilde{c}_V\rho_n}\right] + \frac{\rho_s T\tilde{s}^2}{\rho_n\tilde{c}_V}\left(\frac{\partial P}{\partial\rho}\right)_T = 0$$

...Landau two-fluid model predicts two sound velocities:

$$c_{1,2} = \left[\frac{v_A^2 + v_L^2}{2} \pm \sqrt{\left(\frac{v_A^2 + v_L^2}{2}\right)^2 - v_L^2 v_T^2}\right]^{1/2}$$

$$v_A = \sqrt{\left(\frac{\partial P}{\partial\rho}\right)_{\tilde{s}}}, \quad v_T = \sqrt{\left(\frac{\partial P}{\partial\rho}\right)_T}, \quad v_L = \sqrt{\frac{\rho_s T\tilde{s}^2}{\rho_n\tilde{c}_V}}$$

(adiabatic, isothermal, Landau velocities)

Hydrodynamic modes (sound waves)

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The sound velocities are determined by:

— thermodynamics

— superfluid density

[Landau J. Phys. (USSR) **5**, 71 (1941)]

Hydrodynamic modes and BKT physics

Landau biquadratic equation of sound:

$$c^4 - c^2\left[\left(\frac{\partial P}{\partial \rho}\right)_{\tilde{s}} + \frac{T\tilde{s}^2\rho_s}{\tilde{c}_V\rho_n}\right] + \frac{\rho_s T\tilde{s}^2}{\rho_n\tilde{c}_V}\left(\frac{\partial P}{\partial \rho}\right)_T = 0$$

$$c_{1,2} = \left[\frac{v_A^2 + v_L^2}{2} \pm \sqrt{\left(\frac{v_A^2 + v_L^2}{2}\right)^2 - v_L^2 v_T^2}\right]^{1/2}$$

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