Low-temperature superfluid density expansion of three- and two-dimensional uniform Bose gases

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Abstract

The study of uniform systems of Bose gases plays an important role in the comprehension of their static and dynamic properties. In this work, we use functional integration to investigate D-dimensional uniform bosonic systems of cold atoms. Extending an approach previously outlined [1] we have derived a one-loop expansion of the number density in terms of the condensate density and of the temperature. In the context of the two-fluid model of a superfluid, we also obtained an expression of the normal fluid density analogous to the one by Lev Landau. The normal density has been used, together with the density expansion, to calculate the superfluid density of the system in the low-temperature limit. We obtained an equation that relates the superfluid density with the condensate density and the temperature for three-dimensional systems, while, for two-dimensional systems, we relate the superfluid density with the number density and the temperature.

1) Functional integration

We consider a D-dimensional Bose gas of identical cold atoms with mass m, described by the complex field $\psi(\vec{r},\tau)$. The partition function is

$$\mathcal{Z} = \int D(\bar\psi,\psi) e^{-\frac{S[\bar\psi,\psi]}{\hbar}},$$

$$S[\bar\psi,\psi] = \int_0^{\beta\hbar} d\tau \int_{L^D} d^D r \bar\psi(\vec{r},\tau)\left(\hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m} - \mu\right)\psi(\vec{r},\tau) + \frac{1}{2}\int d^D r' \, |\psi(\vec{r},\tau)|^2 \, V(\vec{r}-\vec{r}\,') \, |\psi(\vec{r}\,',\tau)|^2$$

The superposition of a normal fluid current with velocity $\vec{v}$ and a superfluid current with velocity $\vec{v}_s$ is described through

$$\partial_\tau \to \partial_\tau - i\vec{v}\cdot\vec{\nabla}, \qquad \psi(\vec{r},\tau) \to e^{i\frac{m\vec{v}_s\cdot\vec{r}}{\hbar}}\psi(\vec{r},\tau)$$

The mean field plus gaussian approximation is given by

$$\psi(\vec{r},\tau) = \psi_0 + \eta(\vec{r},\tau)$$

Performing the functional integration, we obtain the grand potential $\Omega = -\beta^{-1}\ln(\mathcal{Z})$ as

$$\Omega(\mu_e,\psi_0^2,T) = \underbrace{L^D\left(-\mu_e\psi_0^2 + \frac{1}{2}g_0\psi_0^4\right)}_{\boldsymbol{\Omega_0}} + \underbrace{\frac{1}{2}\sum_{\vec{k}} E_{\vec{k}}(\psi_0^2)}_{\boldsymbol{\Omega_g^{(0)}}} + \underbrace{\frac{1}{\beta}\sum_{\vec{k}}\ln\left(1 - e^{-\beta\left(E_{\vec{k}}(\psi_0^2) + \hbar(\vec{v}-\vec{v}_s)\cdot\vec{k}\right)}\right)}_{\boldsymbol{\Omega_g^{(T)}}} \qquad (1)$$

where we define the excitation spectrum

$$E_{\vec{k}}(\psi_0^2) = \sqrt{\left(\frac{\hbar^2k^2}{2m} - \mu_e + g_0\psi_0^2 + \psi_0^2\,V(\vec{k})\right)^2 - \left(\psi_0^2\,V(\vec{k})\right)^2}$$

and the effective chemical potential $\mu_e = \mu - \frac{1}{2}m\vec{v}_s\cdot(\vec{v}_s - 2\vec{v})$.

2) Superfluid density

We calculate the total momentum density of the fluid $\vec{\mathcal{P}}$ as

$$\vec{\mathcal{P}} = \frac{1}{L^D}\frac{\partial\Omega(\mu_e,\psi_0^2,T)}{\partial(-\vec{v})}\bigg|_{\mu_e=g_0n_0} = \vec{\mathcal{P}}_0 + \vec{\mathcal{P}}_g^{(0)} + \vec{\mathcal{P}}_g^{(T)}$$

where, since the grand potential $\Omega(\mu_e,\psi_0^2,T)$ is constituted by the three contributions of Eq. (1), we define $\vec{\mathcal{P}}_0$, $\vec{\mathcal{P}}_g^{(0)}$ and $\vec{\mathcal{P}}_g^{(T)}$ accordingly. In particular, they are given by

- $\vec{\mathcal{P}}_0 = n_0\, m\vec{v}_s$
- $\vec{\mathcal{P}}_g^{(0)} = f_g^{(0)}(n_0)\, m\vec{v}_s$
- $\vec{\mathcal{P}}_g^{(T)} = f_g^{(T)}(n_0)\, m\vec{v}_s + n_n(n_0,T)\, m(\vec{v}-\vec{v}_s)$

and we identify the number density $n\,(n_0,T)$ of the system as a function of the condensate number density n_0 and the temperature T as

$$n\,(n_0,T) = -\frac{1}{L^D}\frac{\partial\Omega(\mu_e,\psi_0^2,T)}{\partial\mu_e}\bigg|_{\mu_e=g_0\psi_0^2=g_0n_0} = n_0 + f_g^{(0)}(n_0) + f_g^{(T)}(n_0)$$

with the gaussian zero-temperature contribution to the number density $f_g^{(0)}(n_0)$ and the gaussian finite-temperature contribution $f_g^{(T)}(n_0)$ given by

- $f_g^{(0)}(n_0) = \frac{1}{2L^D}\sum_{\vec{k}}\frac{1}{E_{\vec{k}}(n_0)}\left(\frac{\hbar^2k^2}{2m} + n_0V(\vec{k})\right)$
- $f_g^{(T)}(n_0) = \frac{1}{L^D}\sum_{\vec{k}}\frac{1}{E_{\vec{k}}(n_0)}\left(\frac{\hbar^2k^2}{2m} + n_0V(\vec{k})\right)\frac{1}{(e^{\beta E_{\vec{k}}(n_0)}-1)}$

and the fluid normal density

$$n_n(n_0,T) = \frac{\beta\hbar^2}{mDL^D}\sum_{\vec{k}} k^2\,\frac{e^{\beta E_{\vec{k}}(n_0)}}{(e^{\beta E_{\vec{k}}(n_0)}-1)^2}$$

Therefore, the total momentum density $\vec{\mathcal{P}}$ is given by

$$\vec{\mathcal{P}} = n_s m\vec{v}_s + n_n\,(n_0,T)\, m\vec{v}$$

where the superfluid density n_s is obtained as a function of the condensate density n_0 and the temperature T, namely

$$n_s = n\,(n_0,T) - n_n\,(n_0,T)$$

which is explicitly calculated choosing the interaction $V(\vec{k})$.

3) Condensate fraction for $V(\vec{k}) = g_0 + g_2k^2$

We implement the number density $n\,(n_0,T)$ calculation for bosons with the finite-range interaction $V(\vec{k}) = g_0 + g_2k^2$. In $D=3$ we obtain the condensate fraction n_0/n as a function of the condensate density n_0 and the temperature. In $D=2$ we calculate the zero-temperature condensate fraction n_0/n, since $f_g^{(T)}(n_0)$, according to Mermin-Wagner theorem, is ultraviolet divergent. In the table we report the weakly-interacting low-temperature limit of these results, but the plot derives from the numerical solution of the general equations obtained.

D	$\mathbf{n_0/n}$ **(weakly-interacting limit, low-temperature regime)**
3	$\frac{n_0}{n} = 1 - \frac{8}{3\sqrt{\pi}}(na_s^3)^{1/2}\left[1 - 24\pi(na_s^3)\,\frac{r_{\text{eff}}}{a_s} + \frac{1}{64}\frac{(k_BT)^2}{(na_s^3)^{2/3}}\left(\frac{m}{\hbar^2n^{2/3}}\right)^2\cdot \left(1 - \frac{(k_BT)^2}{320(na_s^3)^{2/3}}\left(\frac{m}{\hbar^2n^{2/3}}\right)^2\left(1 + 8\pi\frac{r_{\text{eff}}}{a_s}(na_s^3)\right)\right)\right]$
2	$\frac{n_0}{n} = 1 - \frac{1}{\lvert\ln(na_s^2)\rvert} - \frac{2\pi(R/a_s)^2}{\lvert\ln(na_s^2)\rvert^2}\,\ln\left(\frac{8\pi^2\lvert\ln(na_s^2)\rvert}{na_s^2e^{\gamma+2}}\right)$

Table: Condensate fraction n_0/n in the weakly-interacting limit and low-temperature regime.

4) Superfluid fraction for $V(\vec{k}) = g_0 + g_2k^2$

We explicitly implement the superfluid density calculation with the finite-range interaction $V(\vec{k}) = g_0 + g_2k^2$. In $D=3$ we express the superfluid density n_s in the low-temperature limit as a function of the condensate density n_0 and the temperature T, providing an explicit implementation of the Josephson relation, while in $D=2$ we express n_s in terms of the number density n and the temperature T. The table contains n_s/n in the weakly-interacting low-temperature regime in which $n_0 \approx n$, but the plot is made from the numerical solution of the general equation.

D	$\mathbf{n_s/n}$ **(weakly-interacting limit, low-temperature regime)**
3	$\frac{n_s}{n} = 1 - \frac{1}{720\sqrt{\pi}}\left(\frac{m}{\hbar^2n^{2/3}}\right)^4\frac{(k_BT)^4}{(na_s^3)^{5/6}}\left[1 - \frac{5}{32}\left(\frac{m}{\hbar^2n^{2/3}}\right)^2\frac{(k_BT)^2}{(na_s^3)^{2/3}}\cdot\left(1 + 8\pi\frac{r_{\text{eff}}}{a_s}(na_s^3)\right)\right]$
2	$\frac{n_s}{n} = 1 - \frac{3}{32\pi^3}\left(\frac{m}{\hbar^2n}\right)^3\lvert\ln(na_s^2)\rvert^2(k_BT)^3\left[\zeta(3) - \frac{15\zeta(5)}{16\pi^2}\left(\frac{m}{\hbar^2n}\right)^2\left(1 + 4\pi\frac{R^2}{a_s^2}\frac{na_s^2}{\lvert\ln(na_s^2)\rvert}\right)\lvert\ln(na_s^2)\rvert^2(k_BT)^2\right]$

Table: Superfluid fraction n_s/n in the weakly-interacting limit and low-temperature regime.

5) Condensate fraction in $D=2$: a comparison

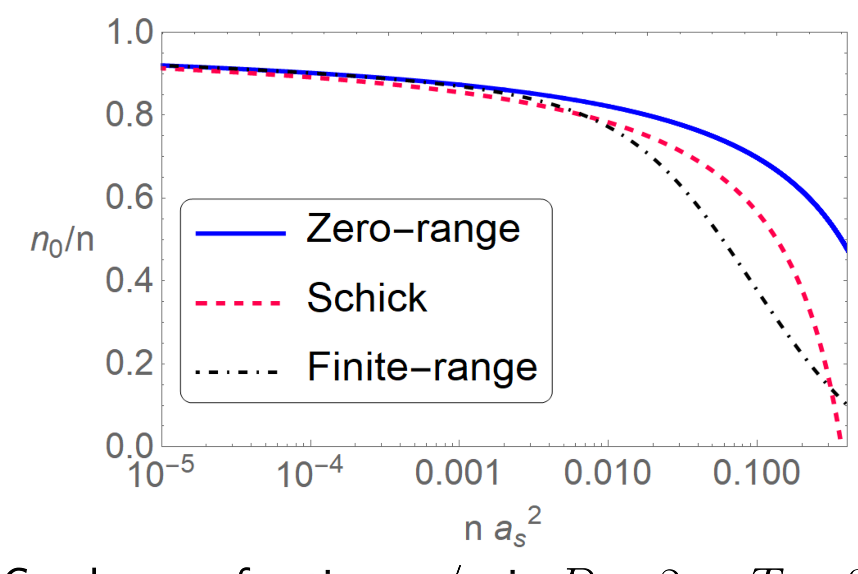

Figure: Condensate fraction n_0/n in $D=2$ at $T=0$, as a function of the gas parameter na_s^2

6) Superfluid fraction in $D=3$

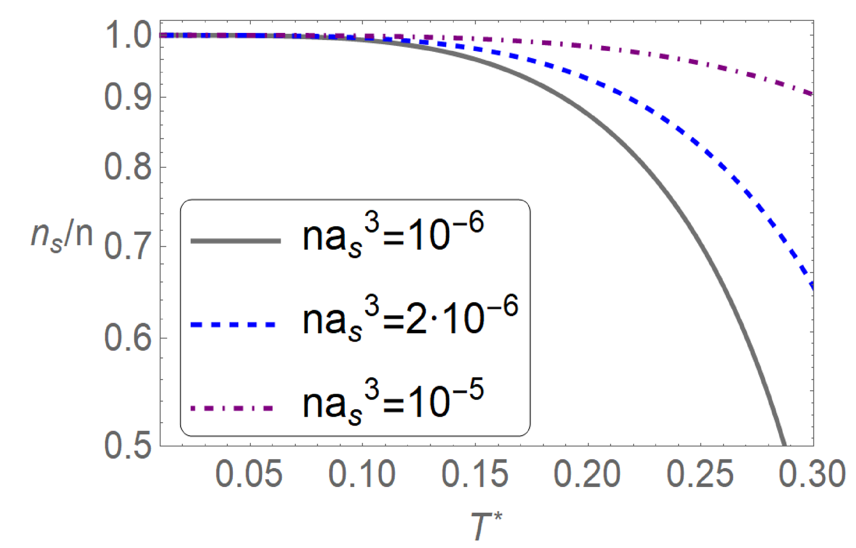

Figure: Superfluid fraction n_s/n in $D=3$ as a function of the rescaled temperature $T^* = k_BT/E_r$, where $E_r = \hbar^2n^{2/3}/m$ is an arbitrary energy scale.

References

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