

# UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”

Corso di Laurea Magistrale in Fisica

Tesi di Laurea

**Bose-Einstein Condensation and Superfluidity**

**in 3D and 2D bosons**

Relatore

Prof. Luca Salasnich

Controrelatore

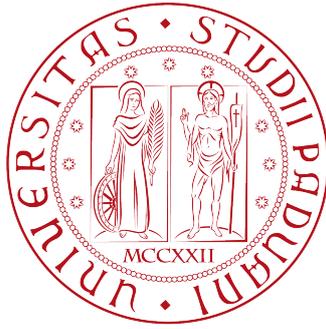
Prof. Francesco Ancilotto

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Anno Accademico 2017/2018





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"In the sciences there need not be a progress of another sort. [...] We are all deeply accustomed to seeing science as the one enterprise that draws constantly nearer to some goal set by nature in advance. But need there be any such goal? [...] For many men the abolition of that teleological kind of evolution was the most significant and least palatable of Darwin's suggestions. The Origin of Species recognized no goal set either by God or nature. Instead, natural selection, operating in the given environment and with the actual organisms presently at hand, was responsible for the gradual but steady emergence of more elaborate, further articulated, and vastly more specialized organisms."

— Thomas S. Kuhn, *The Structure of Scientific Revolutions* [26]



*A Federica*



## **Abstract**

The refinement and the combination of laser cooling and evaporative cooling in dilute systems of alkali atoms lead to the achievement of Bose-Einstein condensation in 1995. Nowadays there is a fruitful interplay between theoretical and experimental results: in this context, the study of uniform Bose gases plays an important role in the comprehension of the static and dynamic properties of a bosonic system. Here we use quantum field theory, within the approach of functional integration, to study an homogeneous system of weakly-interacting bosonic cold atoms. After characterizing their thermodynamical properties, we derive - at a gaussian level - the number density and the superfluid density in three and in two spatial dimensions, expressed as functions of the condensate density and the temperature. Since the equations derived are valid for a generic interaction potential, we implement these formulas for bosons with a zero-range interaction, providing also an extension of these results with the inclusion of a finite-range interaction.



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# Introduction

The study of Bose-Einstein condensation and superfluidity has an ideal origin in the 1924-25 works of Bose and Einstein [9, 14, 15], who predicted the possibility to have a macroscopic occupation of the lowest energy single particle state of the system. While superfluidity in He-II has been experimentally studied since the late thirties [22], the achievement of Bose-Einstein condensation (BEC) has been possible with cold alkali atoms systems only in 1995. In that year condensation has been observed first by a group lead by Carl Wieman and Eric Cornell of the University of Colorado at Boulder [13], but few months later a group lead by Wolfgang Ketterle at the MIT obtained better results, with higher densities [3]. In order to observe quantum degeneracy it was necessary to reach low temperatures, but also low densities, so that the system does not solidify. This has been possible through the combined use of laser cooling techniques and evaporative cooling techniques [23].

Nowadays, the study of cold atoms systems focuses on a large variety of aspects [12] and there is a fruitful interplay between experimental and theoretical results. One of the key aspects of the success of cold atoms systems is constituted by the possibility of tuning the interaction strength between atoms by the means of a Feshbach resonance [11], which provides a very useful tool in the test of our theories.

In the thesis we study, adopting quantum field theory within the formalism of functional integration [2], an homogeneous bosonic system of identical particles in three and in two spatial dimensions. The study of interacting homogeneous condensates [4, 42] is fundamental to understand the basic physical behavior of a BEC and began from the quantum depletion of Bogoliubov [8] in  $D = 3$  and the papers on the one-loop corrections of the energy of Lee and Yang [30], Lee Yang and Huang [31] in  $D = 2$ . In two dimensions Schick found the equation of state for a uniform repulsive gas [46], improved by Popov [41], while Mora and Castin obtained the beyond-gaussian grand potential [37]. In this work, we reproduce some of these results in an unified framework. Moreover, for bosons with a zero-range and a finite-range interaction, we find an explicit implementation of the Josephson relation [21] which links the superfluid density with the condensate density and the temperature.

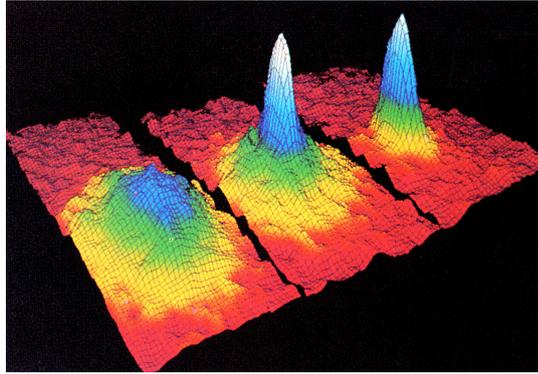


Figure 1: Velocity distribution of a cloud of alkali atoms, obtained after a free expansion of the cloud with imaging techniques. The left picture represents an expanding cloud slightly above the condensation critical temperature, while the central and the right pictures show the occurrence of Bose-Einstein condensation under the critical temperature, represented by the central peak. The figure is obtained from the first observation of Bose-Einstein condensation by Wieman and Cornell group [13].

This thesis is constituted by three main chapters and two appendices.

In the first chapter we will construct the bosonic functional integral and we will calculate the gaussian grand potential of the system, introducing the basic formalism and the techniques which we will employ in the following chapters.

In the second chapter we will derive the thermodynamical properties of an interacting cold atoms system, together with an equation of the system number density expressed as a function of the condensate density and the temperature. We will also calculate the superfluid density of the system following both Lev Landau historical approach and an equivalent microscopic derivation.

In the third chapter we will focus on the study of bosons interacting with a zero-range and with a finite-range interaction, providing an explicit implementation of the results obtained in the second chapter.

In the appendices, we describe how to calculate a summation over Matsubara frequencies and we present a parametrization of the bosonic field alternative to the one used in the first chapter.

# Chapter 1

## Functional integration of the bosonic field

The first chapter begins with the basic formalism of functional integration, from the introduction of second quantization and coherent states, to the calculation of the partition function of the system as the functional integral of the bosonic field. We explicitly perform the gaussian integration of the bosonic field, obtaining the grand potential of the system. In the last section of this chapter we introduce the technique of dimensional regularization, which will be used in the next chapters.

### 1.1 Second quantization and bosonic coherent states

Let us consider a  $D$ -dimensional system of  $N$  spinless bosonic atoms with mass  $m$  interacting with  $V(\vec{r}, \vec{r}')$ . All the properties of the system are enclosed in the first quantization hamiltonian

$$\hat{H} = \sum_{i=1}^N \hat{h}(\vec{r}_i) + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N V(\vec{r}_i, \vec{r}_j) \quad (1.1)$$

where  $\hat{h}(\vec{r}_i)$  is the single particle hamiltonian

$$\hat{h}(\vec{r}_i) = -\frac{\hbar^2 \nabla_i^2}{2m} + U(\vec{r}_i) \quad (1.2)$$

and  $U(\vec{r})$  is the external potential used for trapping the atoms.

The hamiltonian in Eq. (1.1) can be rewritten in the second-quantized form [17]

$$\hat{H} = \sum_{\alpha} \epsilon_{\alpha} \hat{c}_{\alpha}^{\dagger} \hat{c}_{\alpha} + \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} \hat{c}_{\alpha}^{\dagger} \hat{c}_{\beta}^{\dagger} \hat{c}_{\delta} \hat{c}_{\gamma} \quad (1.3)$$

where  $\hat{c}_\alpha^+$  and  $\hat{c}_\alpha$  are the bosonic creation and destruction operators of the Fock state  $|\alpha\rangle$ , and the eigenenergies  $\epsilon_\alpha$

$$\hat{h}|\alpha\rangle = \epsilon_\alpha|\alpha\rangle \quad (1.4)$$

are the eigenvalues of the noninteracting single-particle hamiltonian  $\hat{h}$ . Moreover we have defined the interaction matrix element

$$V_{\alpha\beta\gamma\delta} = \int d^D r d^D r' \phi_\alpha^*(\vec{r}) \phi_\beta^*(\vec{r}') V(\vec{r}, \vec{r}') \phi_\gamma(\vec{r}') \phi_\delta(\vec{r}) \quad (1.5)$$

where  $\phi_\alpha(\vec{r}) = \langle \vec{r} | \alpha \rangle$ .

Defining the field operator  $\hat{\psi}(\vec{r})$  as

$$\hat{\psi}(\vec{r}) = \sum_\alpha \hat{c}_\alpha \phi_\alpha(\vec{r}) \quad (1.6)$$

the second quantization hamiltonian (1.3) can be rewritten in the form

$$\hat{H} = \int d^D r \hat{\psi}^\dagger(\vec{r}) \left( -\frac{\hbar^2 \nabla^2}{2m} + U(\vec{r}) \right) \hat{\psi}(\vec{r}) + \frac{1}{2} \int d^D r d^D r' \hat{\psi}^\dagger(\vec{r}) \hat{\psi}^\dagger(\vec{r}') V(\vec{r}, \vec{r}') \hat{\psi}(\vec{r}') \hat{\psi}(\vec{r}) \quad (1.7)$$

In the following we will concentrate on homogeneous bosonic systems, where the external potential is  $U(\vec{r}) = 0$ . Depending on the physical phenomena under examination, this assumption may prove crucial to obtain analytical equations for an interacting system.

Let us now introduce the bosonic coherent states: a coherent state  $|\psi\rangle$  can be written in the most general way as the product over all possible single-mode coherent states  $|c_\alpha\rangle$ , namely

$$|\psi\rangle = \prod_\alpha |c_\alpha\rangle \quad (1.8)$$

In particular, a single-mode coherent state  $|c_\alpha\rangle$  is defined as the eigenstate of the destruction operator  $\hat{c}_\alpha$  with eigenvalue  $c_\alpha$ , thus

$$\hat{c}_\alpha |c_\alpha\rangle = c_\alpha |c_\alpha\rangle \quad (1.9)$$

For a bosonic many-particle system one can show that the single-mode coherent state  $|c_\alpha\rangle$  can be written as a superposition of all the possible  $n$  bosons Fock states, i.e.

$$|c_\alpha\rangle = e^{-|c_\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\hat{c}_\alpha)^n}{\sqrt{n_\alpha}} |n_\alpha\rangle \quad (1.10)$$

Remembering that the  $n$ -bosons Fock state is given by  $|n_\alpha\rangle = \frac{(\hat{c}_\alpha^+)^n}{\sqrt{n_\alpha}} |0\rangle$ , we can rewrite the single-mode coherent state as

$$|c_\alpha\rangle = e^{-|c_\alpha|^2/2} e^{\hat{c}_\alpha \hat{c}_\alpha^+} |0\rangle \quad (1.11)$$

Finally, substituting this into equation (1.8) we explicitly obtain the expression of the coherent state

$$|\psi\rangle = e^{-\sum_{\alpha} |c_{\alpha}|^2/2} e^{\sum_{\alpha} \hat{c}_{\alpha} c_{\alpha}^{\dagger}} |0\rangle \quad (1.12)$$

Since the single-mode coherent states are normalized  $\langle\psi_i|\psi_i\rangle = 1$ , it follows that the coherent state is also normalized, namely  $\langle\psi|\psi\rangle = 1$ . Nevertheless one should also consider that neither single-mode coherent states, nor coherent states are orthonormal. In particular the overlap between the coherent states  $|\psi\rangle = \prod_{\alpha} |c_{\alpha}\rangle$  and  $|\phi\rangle = \prod_{\alpha} |d_{\alpha}\rangle$  reads

$$\langle\psi|\phi\rangle = e^{-\frac{1}{2}\sum_{\alpha} [\bar{c}_{\alpha}(c_{\alpha}-d_{\alpha}) - (\bar{c}_{\alpha}-\bar{d}_{\alpha})d_{\alpha}]} \quad (1.13)$$

where  $\bar{c}_{\alpha}$  denotes the complex conjugate of  $c_{\alpha}$  and  $\bar{d}_{\alpha}$  the complex conjugate of  $d_{\alpha}$ .

Since a single-mode coherent state constitutes a complete base of the bosonic Fock space, we can write Dirac's completeness relation as

$$\mathbb{1} = \int \frac{d\bar{c}_{\alpha} dc_{\alpha}}{2\pi i} |c_{\alpha}\rangle \langle c_{\alpha}| \quad (1.14)$$

This property induces a more general completeness relation, written in terms of the coherent state  $|\psi\rangle$ , as

$$\mathbb{1} = \int d[\bar{\psi}, \psi] |\psi\rangle \langle\psi| = \int \prod_{\alpha} \frac{d\bar{c}_{\alpha} dc_{\alpha}}{2\pi i} |c_{\alpha}\rangle \langle c_{\alpha}| \quad (1.15)$$

The coherent states provide a connection between first and second quantization [44]. One can show that the eigenstate of the field operator  $\hat{\psi}(\vec{r})$  in second quantization is the Schrödinger field  $\psi(\vec{r})$ , namely

$$\hat{\psi}(\vec{r}) |\psi\rangle = \psi(\vec{r}) |\psi\rangle \quad (1.16)$$

which shows that the Schrödinger field can be written as

$$\psi(\vec{r}) = \sum_{\alpha} c_{\alpha} \phi_{\alpha}(\vec{r}) \quad (1.17)$$

This connection shows that the Schrödinger field  $\psi(\vec{r})$  and  $c_{\alpha}$  are the same field in reciprocal space, so that we can also write the completeness relation

$$\mathbb{1} = \int \prod_{\vec{r}} \frac{d\bar{\psi}(\vec{r}) d\psi(\vec{r})}{2\pi i} |\psi\rangle \langle\psi| \quad (1.18)$$

Moreover we find that the superposition of two coherent states is

$$\langle\psi|\phi\rangle = e^{-\frac{1}{2}\int d^D r [\bar{\psi}(\vec{r})(\psi(\vec{r})-\phi(\vec{r})) - (\bar{\psi}(\vec{r})-\bar{\phi}(\vec{r}))\phi(\vec{r})]} \quad (1.19)$$

The property (1.18) is fundamental for calculating the trace of a quantum mechanical operator. Given a generic operator  $\hat{A}$  the trace operation  $\text{Tr}[\hat{A}]$  is defined as

$$\text{Tr}[\hat{A}] = \sum_n \langle n | \hat{A} | n \rangle \quad (1.20)$$

where  $|n\rangle$  is a complete set of Hilbert space states. Inserting the identity (1.18) inside the bracket of the previous equation we can write

$$\text{Tr}[\hat{A}] = \sum_n \int \prod_{\vec{r}} \frac{d\bar{\psi}(\vec{r}) d\psi(\vec{r})}{2\pi i} \langle n | \psi \rangle \langle \psi | \hat{A} | n \rangle \quad (1.21)$$

Since we are working with bosonic coherent states, we can exchange the brackets inside the integral without any effect on them and, using again the completeness relation  $\mathbb{1} = \sum_n |n\rangle \langle n|$  to eliminate the states  $|n\rangle$ , we get

$$\text{Tr}[\hat{A}] = \int \prod_{\vec{r}} \frac{\bar{\psi}(\vec{r}) d\psi(\vec{r})}{2\pi i} \langle \psi | \hat{A} | \psi \rangle \quad (1.22)$$

which gives the trace operation as an integral over all the possible eigenvalues of the single mode coherent-states.

## 1.2 Construction of the functional integral

The thermodynamical properties of a system of bosonic particles can be described calculating the grand canonical partition function

$$\mathcal{Z} = \text{Tr}[e^{-\beta(\hat{H}-\mu\hat{N})}] \quad (1.23)$$

where  $\mu$  is the chemical potential of the system,  $\hat{H}$  is the hamiltonian of Eq. (1.7) and

$$\hat{N} = \int d^D r \hat{\psi}^\dagger(\vec{r})\hat{\psi}(\vec{r}) \quad (1.24)$$

is the number operator in second quantization.

Using the trace definition of Eq. (1.22) we calculate the grand canonical partition function  $\mathcal{Z}$  as

$$\mathcal{Z} = \int d[\bar{\psi}_0, \psi_0] \langle \psi_0 | e^{-\beta(\hat{H}-\mu\hat{N})} | \psi_0 \rangle \quad (1.25)$$

The main problem of this expression is that the expectation value  $\langle \psi_0 | e^{-\beta(\hat{H}-\mu\hat{N})} | \psi_0 \rangle$  cannot be calculated directly. This issue can be solved by formally writing  $\beta$  in the exponential as

$$\beta = \frac{M\Delta\tau}{\hbar} \quad (1.26)$$

where  $M$  is a large integer and  $\Delta\tau$  is a small imaginary time interval. As we will see during the functional integral construction, this definition allows us to formally introduce the time dependence of the bosonic field in the equations. Another advantage is the possibility to express the exponential as the product of  $M$  equal exponentials  $e^{-\frac{\Delta\tau}{\hbar}(\hat{H}-\mu\hat{N})}$ . In particular, we calculate the integrand of Eq. (1.25) introducing a total number of  $M-1$  identities in the form of Eq. (1.18) between each of the  $M$  exponentials, so that

$$\begin{aligned} \langle \psi_0 | e^{-\beta(\hat{H}-\mu\hat{N})} | \psi_0 \rangle &= \langle \psi_0 | e^{-\frac{\Delta\tau}{\hbar}(\hat{H}-\mu\hat{N})} \dots e^{-\frac{\Delta\tau}{\hbar}(\hat{H}-\mu\hat{N})} | \psi_0 \rangle = \\ &= \int \prod_{j=1}^{M-1} d[\bar{\psi}_j, \psi_j] \prod_{j=1}^M \langle \psi_j | e^{-\frac{\Delta\tau}{\hbar}(\hat{H}-\mu\hat{N})} | \psi_{j-1} \rangle \end{aligned} \quad (1.27)$$

The idea behind this procedure is to take the limit  $M \rightarrow \infty$ , for which the imaginary time interval  $\Delta\tau$  becomes infinitesimal. In this limit, one can approximate the bracket in the third member of Eq. (1.27) by making a negligible error, according to the relation

$$\langle \psi_j | e^{-\frac{\Delta\tau}{\hbar}(\hat{H}-\mu\hat{N})} | \psi_{j-1} \rangle = \langle \psi_j | \psi_{j-1} \rangle e^{-\frac{\Delta\tau}{\hbar}(E[\bar{\psi}_j, \psi_j] - \mu N[\bar{\psi}_j, \psi_j])} + o((\Delta\tau)^2) \quad (1.28)$$

where  $E[\bar{\psi}_j, \psi_j]$  is the energy eigenvalue

$$\begin{aligned} E[\bar{\psi}_j, \psi_j] &= \int d^D r \bar{\psi}_j(\vec{r}) \left( -\frac{\hbar^2 \nabla^2}{2m} + U(\vec{r}) \right) \psi_j(\vec{r}) + \\ &\quad \frac{1}{2} \int d^D r d^D r' \bar{\psi}_j(\vec{r}) \bar{\psi}_j(\vec{r}') V(\vec{r}, \vec{r}') \psi_j(\vec{r}') \psi_j(\vec{r}) \end{aligned} \quad (1.29)$$

and  $N[\bar{\psi}_j, \psi_j]$  is the number operator eigenvalue

$$N[\bar{\psi}_j, \psi_j] = \int d^D r \bar{\psi}_j(\vec{r}) \psi_j(\vec{r}) \quad (1.30)$$

Due to the property of Eq. (1.19) we can express the overlap between the coherent states as

$$\langle \psi_j | \psi_{j-1} \rangle = e^{-\frac{\Delta\tau}{2} \int d^D r [\bar{\psi}_j(\vec{r}) \frac{(\psi_j(\vec{r}) - \psi_{j-1}(\vec{r}))}{\Delta\tau} - \frac{(\bar{\psi}_j(\vec{r}) - \bar{\psi}_{j-1}(\vec{r}))}{\Delta\tau} \psi_j(\vec{r})]} \quad (1.31)$$

Formally, in the  $M \rightarrow \infty$  limit we introduce the continuous imaginary time variable  $\tau = j\Delta\tau$ , identifying the bosonic complex field  $\psi(\vec{r}, \tau)$  as

$$\psi(\vec{r}, \tau) = \psi_j(\vec{r}) \quad (1.32)$$

The imaginary time  $\tau$  definition also allows us to define the time derivatives

$$\partial_\tau \psi(\vec{r}, \tau) = \frac{(\psi_j(\vec{r}) - \psi_{j-1}(\vec{r}))}{\Delta\tau} \quad (1.33)$$

Now, we take the definition of Eq. (1.33) together with Eqs. (1.29), (1.30) and (1.31) and insert them in the partition function (1.25). After some calculations one obtains the grand canonical partition function  $\mathcal{Z}$  expressed as the functional integral of the bosonic complex field  $\psi(\vec{r}, \tau)$

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}, \psi] e^{-\frac{S[\bar{\psi}, \psi]}{\hbar}} \quad (1.34)$$

where

$$S[\bar{\psi}, \psi] = \int_0^{\beta\hbar} d\tau \int_{L^D} d^D r \mathcal{L}(\bar{\psi}, \psi) \quad (1.35)$$

is the euclidean (with imaginary time) action and

$$\mathcal{L} = \bar{\psi}(\vec{r}, \tau) \left( \hbar \partial_\tau - \frac{\hbar^2 \nabla^2}{2m} - \mu \right) \psi(\vec{r}, \tau) + \frac{1}{2} \int d^D r' \bar{\psi}(\vec{r}, \tau) \bar{\psi}(\vec{r}', \tau) V(\vec{r}, \vec{r}') \psi(\vec{r}', \tau) \psi(\vec{r}, \tau) \quad (1.36)$$

is the lagrangian density of the system.

The knowledge of the partition function  $\mathcal{Z}$  is fundamental to obtain the thermodynamics of the system, that can be derived from the grand potential

$$\Omega = -\frac{1}{\beta} \ln(\mathcal{Z}) \quad (1.37)$$

which we will be explicitly calculated in the next section.

Before the end of this section, let us briefly discuss the interaction potential  $V(\vec{r}, \vec{r}')$  between bosons. In order to maintain a general approach we do not specify the form of

the two-body interaction  $V(\vec{r}, \vec{r}')$ , however, we will assume that the interaction potential depends only on the distance  $\vec{r} - \vec{r}'$  between atoms

$$V(\vec{r}, \vec{r}') = V(\vec{r} - \vec{r}') \quad (1.38)$$

This simplified dependence allows us to express the interaction potential as the Fourier series

$$V(\vec{r} - \vec{r}') = \frac{1}{L^D} \sum_{\vec{k}} e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} V(\vec{k}) \quad (1.39)$$

where the Fourier components  $V(\vec{k})$  are given by

$$V(\vec{k}) = \int d^D r e^{-i\vec{k} \cdot (\vec{r} - \vec{r}')} V(\vec{r} - \vec{r}') \quad (1.40)$$

In the next section we will introduce the phenomenon of the spontaneous symmetry breaking of the global U(1) symmetry of the lagrangian (1.36). This phenomenological consideration will allow us to calculate the partition function  $\mathcal{Z}$  as the functional integral of the field fluctuations around the uniform field configuration of the broken symmetry phase. In particular, we will obtain a gaussian (one-loop) expression for the grand potential (1.37) of the bosonic system.

### 1.3 Functional integration of the bosonic field

In this section we perform the functional integration of the partition function  $\mathcal{Z}$  in order to obtain the grand potential  $\Omega$  of the bosonic cold atoms system.

The grand canonical partition function  $\mathcal{Z}$ , in the form the functional integral of Eq. (1.34), can be thought as a way to weight over all the possible configurations of the system, given by the complex field  $\psi(\vec{r}, \tau)$ . It is interesting to notice that there are different field configurations which produce the same lagrangian (1.36), i.e. that give the same contribution to the partition function. In particular, let us consider the following phase-shifted complex field  $\psi'(\vec{r}, \tau)$  given by

$$\psi'(\vec{r}, \tau) = e^{i\alpha} \psi(\vec{r}, \tau), \quad \alpha \in \mathbb{R} \quad (1.41)$$

where  $\alpha$  is a real constant. We see that the lagrangian (1.36) is invariant under this transformation, namely

$$\mathcal{L}(\bar{\psi}', \psi') = \mathcal{L}(\bar{\psi}, \psi) \quad (1.42)$$

One usually describe this property by saying that the lagrangian is invariant under the U(1) global phase symmetry. For this reason, under the hypothesis that the global phase symmetry is also respected by the ground state of the system, we must conclude that a field configuration  $\psi(\vec{r}, \tau)$  and its opposite  $-\psi(\vec{r}, \tau) = e^{i\pi} \psi(\vec{r}, \tau)$  are equally probable and that the expectation value of the bosonic field must be zero: this means that, on average, the system does not contain any particles. The problem can be solved by introducing the concept of spontaneous symmetry breaking: the idea is that the symmetry of the lagrangian is not shared by the ground state of the system, which acquires a non-zero expectation value. The crucial point is that the symmetry is realized because every broken-symmetry ground state can be realized with the same probability, but the system cannot spontaneously and instantaneously go from one ground state to another equivalent because of the infinite energy that this operation would cost in the thermodynamic limit. We include this fundamental phenomenological aspect in our theory by using the bosonic field parametrization

$$\psi(\vec{r}, \tau) = \psi_0 + \eta(\vec{r}, \tau) \quad (1.43)$$

where  $\eta(\vec{r}, \tau)$  is the complex field describing the fluctuation around the real, uniform in space and constant in time field configuration  $\psi_0$ , assumed by the system in the broken-symmetry phase.

Substituting the expressions above in the lagrangian (1.36) we obtain

$$\mathcal{L} = -\mu\psi_0^2 + \bar{\eta}(\vec{r}, \tau) \left( \hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m} - \mu \right) \eta(\vec{r}, \tau) - \mu\psi_0(\bar{\eta}(\vec{r}, \tau) + \eta(\vec{r}, \tau)) + \mathcal{L}_{\text{int}} \quad (1.44)$$

with

$$\begin{aligned} \mathcal{L}_{\text{int}} = \frac{1}{2} \int d^D r' V(\vec{r} - \vec{r}') & (\psi_0^4 + \psi_0^3(\bar{\eta}(\vec{r}', \tau) + \bar{\eta}(\vec{r}, \tau) + \eta(\vec{r}', \tau) + \eta(\vec{r}, \tau)) + \\ & \psi_0^2(\bar{\eta}(\vec{r}', \tau)\bar{\eta}(\vec{r}, \tau) + \bar{\eta}(\vec{r}', \tau)\eta(\vec{r}, \tau) + \bar{\eta}(\vec{r}', \tau)\eta(\vec{r}', \tau) + \\ & \bar{\eta}(\vec{r}, \tau)\eta(\vec{r}, \tau) + \bar{\eta}(\vec{r}, \tau)\eta(\vec{r}', \tau) + \eta(\vec{r}, \tau)\eta(\vec{r}', \tau))) \end{aligned} \quad (1.45)$$

This lagrangian has been calculated neglecting terms in  $\eta(\vec{r}, \tau)$  and  $\bar{\eta}(\vec{r}, \tau)$  greater than second order, namely working under the hypothesis that the fluctuations around the mean field configuration  $\psi_0$  of the system are small and noninteracting with each other. One usually refer to this approach as the gaussian, or one-loop, approximation. In the lagrangian (1.44) we have also omitted some terms containing time derivatives, which are zero due to the imaginary time periodicity of the fields.

Given the lagrangian, the action (1.35) can be written as

$$\begin{aligned} S[\bar{\eta}, \eta] = \beta\hbar L^D \mu \psi_0^2 + \int_0^{\beta\hbar} d\tau \int_{L^D} d^D r \left[ \bar{\eta}(\vec{r}, \tau) \left( \hbar \partial_\tau - \frac{\hbar^2 \nabla^2}{2m} - \mu \right) \eta(\vec{r}, \tau) - \right. \\ \left. \mu \psi_0 (\bar{\eta}(\vec{r}, \tau) + \eta(\vec{r}, \tau)) + S_{\text{int}}[\bar{\eta}, \eta] \right] \end{aligned} \quad (1.46)$$

with

$$S_{\text{int}}[\bar{\eta}, \eta] = \int_0^{\beta\hbar} d\tau \int_{L^D} d^D r \mathcal{L}_{\text{int}}(\bar{\eta}, \eta) \quad (1.47)$$

We now express the action of the system in the Fourier space, where the functional field integration can be performed in an explicit way. This can be done introducing the Fourier series of the fluctuation fields, using the convention

$$\begin{aligned} \eta(\vec{r}, \tau) &= \frac{1}{\sqrt{L^D}} \sum_{\vec{k}, \omega_n} e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_n \tau} \eta(\vec{k}, \omega_n), \\ \bar{\eta}(\vec{r}, \tau) &= \frac{1}{\sqrt{L^D}} \sum_{\vec{k}, \omega_n} e^{-i\vec{k}\cdot\vec{r}} e^{i\omega_n \tau} \bar{\eta}(\vec{k}, \omega_n), \\ \eta(\vec{k}, \omega_n) &= \frac{1}{\beta\hbar \sqrt{L^D}} \int_0^{\beta\hbar} d\tau \int_{L^D} d^D r e^{-i\vec{k}\cdot\vec{r}} e^{i\omega_n \tau} \eta(\vec{r}, \tau), \\ \bar{\eta}(\vec{k}, \omega_n) &= \frac{1}{\beta\hbar \sqrt{L^D}} \int_0^{\beta\hbar} d\tau \int_{L^D} d^D r e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_n \tau} \bar{\eta}(\vec{r}, \tau) \end{aligned} \quad (1.48)$$

Although it will not be written explicitly for simplicity of notation, the sum over  $\vec{k}$  does not contain the zero-momentum state, since the homogeneous Fourier component of the field has already been set apart with the parametrization (1.43) of the bosonic field.

One can easily obtain the dimension of the Fourier components of the fluctuation field using dimensional analysis

$$[\eta(k)] = [L^D] \quad [\bar{\eta}(k)] = [L^D]$$

where  $k = (\vec{k}, \omega_n)$  denotes the  $D+1$  wavevectors and  $\omega_n = \frac{2\pi n}{\beta\hbar}$  are the bosonic Matsubara frequencies [2].

Another important property for calculating the Fourier transform of the action  $S[\bar{\eta}, \eta]$  is the  $D + 1$ -dimensional Fourier representation of the delta function

$$\delta(\vec{k} - \vec{k}') \delta(\omega_n - \omega_{n'}) = \int_{L^D} \frac{d^D r}{L^D} e^{-i(\vec{k} - \vec{k}') \cdot \vec{r}} \int_0^{\beta\hbar} \frac{d\tau}{\beta\hbar} e^{i(\omega_n - \omega_{n'})\tau} \quad (1.49)$$

We substitute the Fourier components of Eq. (1.48) in the action (1.47) and use the delta function definition (1.49) to integrate on  $\vec{r}$  and  $\tau$  coordinates, obtaining the action in the Fourier space

$$\begin{aligned} S[\bar{\eta}(k), \eta(k)] = & -\beta\hbar L^D \mu \psi_0^2 + \\ & \beta\hbar \sum_{\vec{k}\omega_n} \left[ \bar{\eta}(k) \left( -i\hbar\omega_n + \frac{\hbar^2 k^2}{2m} - \mu + \frac{\psi_0^2}{2} (V(\vec{k}) + V(-\vec{k})) \right) \eta(k) \right] + \\ & \beta\hbar \sum_{\vec{k}\omega_n} \left[ -\mu\psi_0 \sqrt{L^D} (\bar{\eta}(k) + \eta(k)) \delta(\vec{k}) \delta(\omega_n) + \frac{1}{2} \psi_0^4 L^D V(\vec{k}) \delta(\vec{k}) + \right. \\ & \left. + \psi_0^2 V(\vec{k} = 0) \bar{\eta}(k) \eta(k) + \frac{1}{2} \psi_0^2 V(\vec{k}) (\bar{\eta}(k) \bar{\eta}(-k) + \eta(k) \eta(-k)) \right] \end{aligned} \quad (1.50)$$

In this action appears the zero-wavevector Fourier component of the interaction potential  $V(\vec{k} = 0)$ , which is usually defined as the zero-range (or contact) interaction strength  $g_0$ , according to

$$g_0 = V(\vec{k} = 0) = \int d^D r V(\vec{r}) \quad (1.51)$$

Taking into account this definition, we identify the wavevector-independent part of the action (1.50) as the action of homogeneous system

$$S_0 = \beta\hbar L^D \left( -\mu\psi_0^2 + \frac{1}{2} g_0 \psi_0^4 \right) \quad (1.52)$$

Moreover we consider that, since the sum over wavevectors  $\vec{k}$  in Eq. (1.50) does not include the zero component, the linear terms in  $\bar{\eta}(k)$  and  $\eta(k)$  can be deleted due to the presence of the  $D + 1$ -dimensional delta function, leading to the action

$$\begin{aligned} S[\bar{\eta}(k), \eta(k)] = & S_0 + \\ & \beta\hbar \sum_{\vec{k}\omega_n} \left[ \bar{\eta}(k) \left( -i\hbar\omega_n + \frac{\hbar^2 k^2}{2m} - \mu + g_0 \psi_0^2 + \frac{\psi_0^2}{2} (V(\vec{k}) + V(-\vec{k})) \right) \eta(k) + \right. \\ & \left. \frac{1}{2} \psi_0^2 V(\vec{k}) (\bar{\eta}(k) \bar{\eta}(-k) + \eta(k) \eta(-k)) \right] \end{aligned} \quad (1.53)$$

We now perform the gaussian functional integral of the complex fluctuation field  $\eta(k)$  considering that the action  $S[\bar{\eta}(k), \eta(k)]$  can be rewritten in the matrix form

$$S[\bar{\eta}(k), \eta(k)] = S_0 + \frac{\hbar}{2} \sum_{\vec{k}, \omega_n} (\bar{\eta}(k) \quad \eta(-k)) \mathbf{M}(k) \begin{pmatrix} \eta(k) \\ \bar{\eta}(-k) \end{pmatrix} \quad (1.54)$$

where the  $2 \times 2$  matrix  $\mathbf{M}(k)$  is given by

$$\mathbf{M} = \beta \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \quad (1.55)$$

with

$$\begin{aligned} \mathbf{M}_{11} &= -i\hbar\omega_n + \frac{\hbar^2 k^2}{2m} - \mu + g_0 \psi_0^2 + \frac{\psi_0^2}{2} (V(\vec{k}) + V(-\vec{k})) \\ \mathbf{M}_{22} &= +i\hbar\omega_n + \frac{\hbar^2 k^2}{2m} - \mu + g_0 \psi_0^2 + \frac{\psi_0^2}{2} (V(\vec{k}) + V(-\vec{k})) \\ \mathbf{M}_{12} &= \mathbf{M}_{21} = \psi_0^2 V(\vec{k}) \end{aligned} \quad (1.56)$$

One should also remember that, because of the field parametrization (1.43) the measure of the functional integral must change. Considering that the functional integral can be thought as the limit for  $M \rightarrow \infty$  time steps of the argument of the partition function 1.27, we may extend the usual change of real variables in this "continuum measure" case. Doing so, the measure of the integral becomes

$$\begin{aligned} \int \mathcal{D}[\bar{\psi}, \psi] &= \int \prod_{(\vec{r}, \tau)} \frac{d(\bar{\psi}(\vec{r}, \tau), \psi(\vec{r}, \tau))}{2\pi i} = \int \prod_{(\vec{r}, \tau)} \frac{d\bar{\eta}(\vec{r}, \tau) d\eta(\vec{r}, \tau)}{2\pi i} = \\ &= \int \prod_{\substack{\vec{k}, \omega_n \\ k_z > 0}} \frac{d\bar{\eta}(\vec{k}, \omega_n) d\eta(\vec{k}, \omega_n) d\bar{\eta}(-\vec{k}, -\omega_n) d\eta(-\vec{k}, -\omega_n)}{(2\pi i L^D)^2} \end{aligned} \quad (1.57)$$

where the  $L^D$  factor at the denominator in the last term has been introduced for dimensional reasons (and because of the Fourier series convention). Moreover, we must consider that  $\eta(\vec{k}, \omega_n)$  and  $\eta(-\vec{k}, -\omega_n)$  are not independent integration variables: to prevent overcounting the fields we imposed the restriction  $k_z > 0$  in the functional integral measure.

Substituting the measure (1.57) and the action (1.53) in the equation (1.34) we see that the grand canonical partition function  $\mathcal{Z}$  can be expressed as the product

$$\mathcal{Z} = \mathcal{Z}_0 \cdot \mathcal{Z}_g \quad (1.58)$$

where

$$\mathcal{Z}_0 = e^{-S_0/\hbar} \quad (1.59)$$

is the partition function of the homogeneous system with action (1.52) and  $\mathcal{Z}_g$  is the gaussian grand canonical partition function which can be easily calculated performing a gaussian functional integration [38], namely

$$\begin{aligned} \mathcal{Z}_g &= \int \prod_{\substack{\vec{k} \omega_n \\ k_z > 0}} \frac{d\bar{\eta}(k) d\eta(k) d\bar{\eta}(-k) d\eta(-k)}{(2\pi i L^D)^2} \exp \left( - \sum_{\substack{\vec{k} \omega_n \\ k_z > 0}} (\bar{\eta}(k) \quad \eta(-k)) \mathbf{M}(k) \begin{pmatrix} \eta(k) \\ \bar{\eta}(-k) \end{pmatrix} \right) = \\ &= \prod_{\substack{\vec{k} \omega_n \\ k_z > 0}} \frac{1}{L^{2D}} [\det \mathbf{M}(k)]^{-1} \end{aligned} \quad (1.60)$$

The gaussian approximation allowed an explicit calculation of the partition function  $\mathcal{Z}$ , which is expressed in Eq. (1.58) as the product of  $\mathcal{Z}_0$  and  $\mathcal{Z}_g$ . It follows that the grand potential, given by  $\Omega = -\beta^{-1} \ln(\mathcal{Z})$ , is expressed as the sum of two main contributions

$$\Omega = \Omega_0 + \Omega_g \quad (1.61)$$

where

$$\Omega_0 = L^D \left( -\mu \psi_0^2 + \frac{1}{2} g_0 \psi_0^4 \right) \quad (1.62)$$

is the zero-temperature grand potential of the uniform system. Regarding the gaussian grand potential  $\Omega_g$ , we calculate it considering that the factor  $L^{2D}$  at the denominator of the grand canonical partition function  $\mathcal{Z}_g$  can be neglected, since it shifts the grand potential of a  $\mu$  independent quantity. We get

$$\Omega_g = -\frac{1}{\beta} \ln(\mathcal{Z}_g) = -\frac{1}{\beta} \sum_{\substack{\vec{k} \omega_n \\ k_z > 0}} \ln[\det \mathbf{M}(k)]^{-1} \quad (1.63)$$

The determinant of the gaussian matrix  $\mathbf{M}(k)$  can be easily calculated, obtaining the gaussian grand potential  $\Omega_g$  as

$$\Omega_g = \frac{1}{2\beta} \sum_{\vec{k} \omega_n} \ln[\beta^2 (\hbar^2 \omega_n^2 + E_{\vec{k}}^2)] \quad (1.64)$$

where we have defined the excitation spectrum of the elementary excitations of the bosonic system as

$$E_{\vec{k}} = \sqrt{\left( \frac{\hbar^2 k^2}{2m} - \mu + g_0 \psi_0^2 + \frac{\psi_0^2}{2} (V(\vec{k}) + V(-\vec{k})) \right)^2 - (\psi_0^2 V(\vec{k}))^2} \quad (1.65)$$

To explicitly calculate  $\Omega_g$  we perform the summation over the Matsubara frequencies  $\omega_n$ . For simplicity, we refer the reader to the Appendix A for the detailed calculation. We get the gaussian grand potential  $\Omega_g$  as the sum of two main terms

$$\Omega_g = \Omega_g^{(0)} + \Omega_g^{(T)} \quad (1.66)$$

where

$$\Omega_g^{(0)} = \frac{1}{2} \sum_{\vec{k}} E_{\vec{k}} \quad (1.67)$$

is the zero-temperature gaussian grand potential, written as the sum of noninteracting bosonic excitations with spectrum  $E_{\vec{k}}$ , while

$$\Omega_g^{(T)} = \frac{1}{\beta} \sum_{\vec{k}} \ln(1 - e^{-\beta E_{\vec{k}}}) \quad (1.68)$$

is the thermal gaussian contribution to the grand potential.

Considering the various contributions of Eqs. (1.62), (1.67) and (1.68), the grand potential  $\Omega$  of the bosonic cold atoms system reads

$$\Omega = L^D \left( -\mu\psi_0^2 + \frac{1}{2}g_0\psi_0^4 \right) + \frac{1}{2} \sum_{\vec{k}} E_{\vec{k}} + \frac{1}{\beta} \sum_{\vec{k}} \ln(1 - e^{-\beta E_{\vec{k}}}) \quad (1.69)$$

One can also consider the thermodynamic limit of the system, in which the sum over the wavevectors  $\vec{k}$  can be substituted with a continuous  $D$ -dimensional integral, following the prescription

$$\frac{1}{L^D} \sum_{\vec{k}} = \int \frac{d^D k}{(2\pi)^D} \quad (1.70)$$

In the thermodynamic limit, we express the grand potential per unit of volume  $\Omega/L^D$  as

$$\frac{\Omega}{L^D} = \left( -\mu\psi_0^2 + \frac{1}{2}g_0\psi_0^4 \right) + \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} E_{\vec{k}} + \frac{1}{\beta} \int \frac{d^D k}{(2\pi)^D} \ln(1 - e^{-\beta E_{\vec{k}}}) \quad (1.71)$$

Finally, we add that the  $D$ -spatial dimensions integrals over the wavevector  $\vec{k}$  are consistently easier to calculate in the case in which the integrand function depends only on the modulus of the wavevector  $|\vec{k}|$ . In fact, one can employ  $D$ -dimensional spherical coordinates (also denoted as  $D$ -dimensional polar coordinates) to express the integral of a generic  $|\vec{k}|$ -dependent integrand function  $F(|\vec{k}|)$  as

$$\int \frac{d^D k}{(2\pi)^D} F(|\vec{k}|) = \frac{S_D}{(2\pi)^D} \int_0^{+\infty} dk k^{D-1} F(k) \quad (1.72)$$

where  $S_D = \frac{2\pi^{D/2}}{\Gamma[D/2]}$  is the surface element of the  $D$ -dimensional sphere, namely the solid angle in  $D$  dimensions. We anticipate that, in the thesis, we will need to explicitly calculate only integrals in which this property holds.

In the next chapter, we will rely on the framework outlined here to derive explicit equations for the system thermodynamics and to characterize its fundamental static and dynamic properties, among which we will concentrate on the condensate fraction and the superfluid density. Before of this, we introduce the procedure and the basic ideas of dimensional regularization, which will be extensively used in the next chapters.

## 1.4 Regularization of divergent integrals

Since the early years of quantum mechanics, the regularization of divergent integrals has been a challenging task, which produced a multitude of different techniques. In this thesis we will need to treat ultraviolet-diverging integrals, which means that they diverge in the high wavevector part of the spectrum. The idea behind the elimination of these degrees of freedom is that high wavevector processes are associated to small spatial scales, namely they are high energy processes. Since we do not pretend that our theory is able to correctly describe these processes and considering that from an experimental point of view we do not observe any divergence, we decide to cut-off this part of the spectrum.

In our framework, an effective way to regularize the diverging integrals is constituted by dimensional regularization, which has been developed by 't Hooft and Veltman [48]. The basic result on which this technique is based is the Veltman conjecture [33]

$$\int_0^{+\infty} dk k^{D-1} (k^2)^{n-1} = 0 \quad n = 0, 1, 2, \dots \quad (1.73)$$

Moreover, when the Veltmann conjecture is not sufficient or cannot be applied, one can achieve the regularization of the divergences employing the Euler gamma and beta functions, defined as

$$\Gamma(z) = \int_0^{+\infty} dt t^{z-1} e^{-t}, \quad \text{Re}(z) > 0 \quad (1.74)$$

$$B(x, y) = \int_0^{+\infty} dt \frac{t^{x-1}}{(1+t)^{x+y}}, \quad \text{Re}(x), \text{Re}(y) > 0 \quad (1.75)$$

The main fundamental property of the gamma function is that

$$\Gamma[z + 1] = z \Gamma[z] \quad (1.76)$$

which can be demonstrated with integration by parts of the gamma definition and constitutes the reason why this function is usually described as an extension of the factorial operation. Using this property, one can extend the domain of the gamma function by analytic continuation to values of  $z$  with  $\text{Re}(z) < 0$ . The same can be done for the beta function, relying on the property

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}, \quad \text{Re}[x], \text{Re}[y] > 0 \quad (1.77)$$

which can be used to extend the definition of the beta function to negative values of its arguments. The basic idea of the dimensional regularization with the beta function is to rewrite the diverging integral in the form of Eq. (1.75) and use the property (1.77) - even outside the correct domain of application given by  $\text{Re}[x], \text{Re}[y] > 0$  - to express the

initial integral in terms of a product of convergent gamma function. It is also useful to define the logarithmic derivative of the gamma function, usually denoted as the digamma function, which is

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (1.78)$$

Essentially, the values of Euler gamma, digamma and beta functions can be obtained applying the relations (1.76), (1.77), with additional details that are given by [24]. Since they allow to obtain all the other main values and because they will prove useful in the next chapters, we only report the values of

$$\Gamma(1/2) = \sqrt{\pi} \quad \Psi(1) = -\gamma \quad (1.79)$$

where  $\gamma \approx 0.55722$  is the Euler-Mascheroni constant.

Despite the dimensional regularization procedure described above can be successfully used to regularize many ultraviolet diverging integrals, the gamma function evaluated for negative integer values, for which it is still infinite, may appear in the calculations. The usual procedure to treat this divergence consists in extending the dimension of the system to the complex value  $\mathcal{D} = D - \varepsilon$  and, after performing the integration, taking the limit  $\varepsilon \rightarrow 0$ . For this approach, a fundamental result is constituted by [24]

$$\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \left[ \frac{1}{\varepsilon} + \Psi(n+1) + \frac{\varepsilon}{2} \left( \frac{\pi^2}{3} + \Psi(n+1)^2 - \Psi'(n+1) \right) + o(\varepsilon^2) \right] \quad (1.80)$$

Moreover, using the dimension  $\mathcal{D} = D - \varepsilon$ , all the  $D$ -dependent physical quantities will acquire a dependence on  $\varepsilon$ : in the  $\varepsilon \rightarrow 0$  limit one can express the Taylor series of a generic coefficient  $x^\varepsilon$  as

$$x^\varepsilon = \exp(\varepsilon \ln(x)) \sim_{\varepsilon \rightarrow 0} 1 + \varepsilon \ln(x) + o(\varepsilon^2) \quad (1.81)$$

The extension to the complex dimension  $\mathcal{D}$  usually allows to perform the analytic integration with the beta function (1.75). However, the divergence of the gamma function for negative integer arguments shows up again when one takes the limit  $\varepsilon \rightarrow 0$ , being enclosed in the  $o(\varepsilon^{-1})$  term. The idea is to expand, using the Eqs. (1.80) and (1.81), all the  $\varepsilon$  dependent functions in terms of small  $\varepsilon$  and, after multiplication of all factors, explicitly delete the diverging  $o(\varepsilon^{-1})$  terms.

This procedure is based on the philosophy [49] that regularization consists in

*"Extracting Finite Information from Infinities"*

which is actually what one does by considering only the  $o(\varepsilon^0)$  terms and deleting the divergent ones.



## Chapter 2

# Statical and dynamical properties of a bosonic system

In the second chapter we apply and extend the fundamental results of the previous one. In the first section we derive general equations for the thermodynamics of a Bose gas, while in the second we express its density  $n$  as a function of the condensate density  $n_0$  and the temperature  $T$ . In the last sections we deal with dynamical properties of a Bose-Einstein condensate, namely we work within the framework of the two-fluid model of a superfluid, to derive the fluid normal density  $n_n$  first in the Lev Landau historical approach, then following and extending a derivation by Adriaan Schakel [45].

### 2.1 Thermodynamic properties

In the previous chapter we obtained the grand potential  $\Omega$  as a function of  $\mu$ ,  $\psi_0$ , and  $T$ . In the broken-symmetry condensate phase, the value of the order parameter  $\psi_0$  is different from zero and constitutes the uniform configuration assumed by the system.

To derive the thermodynamical properties of the system we must fix the value of the order parameter  $\psi_0$ , being able to express it as a function of the chemical potential  $\mu$  and the temperature  $T$ . The most common way to determine the uniform field configuration  $\psi_0$  consists in using an energy variational principle. We impose the saddle point condition, namely we impose that  $\psi_0$  minimizes the uniform grand potential

$$\left( \frac{\partial \Omega_0(\mu, \psi_0)}{\partial \psi_0} \right)_{\mu, T, L^D} = 0 \quad (2.1)$$

which leads to the saddle point condition

$$\psi_0 = \sqrt{\frac{\mu}{g_0}} \quad (2.2)$$

Substituting this value of  $\psi_0$  in the excitation spectrum (1.65), it becomes

$$E_{\vec{k},\text{sp}} = \sqrt{\left(\frac{\hbar^2 k^2}{2m} + \frac{\mu}{2g_0}(V(\vec{k}) + V(-\vec{k}))\right)^2 - \left(\frac{\mu}{g_0}V(\vec{k})\right)^2} \quad (2.3)$$

The determination of the mean field value of  $\psi_0$  in Eq. (2.2) allows us to explicitly calculate the grand potential of the order parameter per unit of volume  $\Omega_0/L^D$ , which, according to Eq. (1.62), reads

$$\frac{\Omega_0}{L^D} = -\frac{\mu^2}{2g_0} \quad (2.4)$$

which does not depend on the specific interaction  $V(\vec{k})$  between bosons, but only on the contact interaction strength  $g_0$ . On the contrary, the gaussian contribution to the grand potential per unit of volume  $\Omega_g/L^D$  depends on the excitation spectrum (2.3). In the thermodynamic limit it is given by Eqs. (1.67) and (1.68)

$$\frac{\Omega_g}{L^D} = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} E_{\vec{k},\text{sp}} + \frac{1}{\beta} \int \frac{d^D k}{(2\pi)^D} \ln\left(1 - e^{-\beta E_{\vec{k},\text{sp}}}\right) \quad (2.5)$$

therefore will give different results depending on the choice of the interaction  $V(\vec{k})$ : in the next chapter we shall implement Eq. (2.5) in three and two spatial dimensions, considering a zero-range interaction or a finite-range effective interaction. Since the first contribution to the Eq. (2.5) is ultraviolet divergent, we will explicitly regularize it employing the dimensional regularization techniques described in section (1.4).

The knowledge of the grand potential  $\Omega$  allows to calculate the main thermodynamical functions of the system. In particular, the pressure can easily be obtained as the opposite of the grand potential per unit of volume  $\Omega/L^D$ , according to the relation

$$P = -\frac{\Omega}{L^D} \quad (2.6)$$

We also remind the definition of the entropy  $S$ , which, for a system at the thermodynamical equilibrium, reads

$$S = -\left(\frac{\partial\Omega}{\partial T}\right)_{\mu, L^D} \quad (2.7)$$

In particular, we now discuss the number density  $n$  of the system, which is given by the thermodynamical relation

$$n = -\frac{1}{L^D} \left(\frac{\partial\Omega}{\partial\mu}\right)_{T, L^D} \quad (2.8)$$

Substituting the grand potential contributions given by Eqs. (2.4) and (2.5) we find

$$n = \frac{\mu}{g_0} - \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \frac{\partial E_{\vec{k},\text{sp}}}{\partial\mu} - \frac{1}{\beta} \int \frac{d^D k}{(2\pi)^D} \frac{\partial E_{\vec{k},\text{sp}}}{\partial\mu} \frac{1}{e^{\beta E_{\vec{k},\text{sp}}} - 1} \quad (2.9)$$

Notice that the first term does not explicitly depend on the form of the interaction  $V(\vec{k})$  while, as for the grand potential, the gaussian corrections given by the second and the third term of Eq. (2.9), depend on the excitation spectrum  $E_{\vec{k},\text{sp}}$ . To perform an explicit calculation, depending on the spatial dimension  $D$  and since a regularization procedure of the integrals is needed, it is either convenient to calculate the grand potential (2.5) and then derive it with respect to  $\mu$  to get the number density, or to use directly the expression (2.9).

As an important result of this thesis, we will derive an equation for the number density  $n$  expressed as a function of the condensate density  $n_0$  and the temperature  $T$ . However, we will not employ the number density of Eq. (2.8), in which we stress that the derivative of the grand potential  $\Omega$  with respect to the chemical potential  $\mu$  is taken after having substituted the saddle point condition (2.2). We remind this detail because in the next section we will be following another procedure.

In a system with a fixed number density  $n$ , it may be useful to determine the chemical potential  $\mu$  as a function of the number density  $n$  and the temperature  $T$

$$\mu = \mu(n, T) \tag{2.10}$$

From a mathematical point of view this operation is denoted as the Legendre transform of the grand potential  $\Omega$ . Employing a computational approach, one can numerically calculate the chemical potential  $\mu$  from the system number density of Eq. (2.9), since it constitutes an implicit expression in the form (2.10).

In the next section we will introduce the basic concepts of Bose-Einstein condensation. We will obtain an equation for the system density  $n$  expressed as a function of the condensate density  $n_0$  and the temperature  $T$ . This equation will be employed, in the next chapter, to explicitly calculate the condensate fraction  $n_0/n$  for three- and two-dimensional bosons interacting with a zero-range and a finite-range interaction.

## 2.2 Quantum depletion and thermal condensate fraction

Given a system of bosonic non-interacting atoms at the thermodynamic equilibrium with temperature  $T$ , we define the thermal de Broglie wavelength as

$$\Lambda = \frac{h}{\sqrt{2\pi m k_B T}} \quad (2.11)$$

The thermal wavelength  $\Lambda$  can be thought as the cube root of the quantum volume associated to an atom, therefore represents the length scale at which the particles overlap and their intrinsic quantum nature must be taken into account. For this reason,  $\Lambda$  is usually compared with the mean interparticle spacing, which for a system with number density  $n$ , reads

$$d = n^{1/3} \quad (2.12)$$

Notice that  $\Lambda$  increases by decreasing the temperature: while classical physics can be successfully used to describe many properties of dilute systems of weakly-interacting gases at high temperature, a quantum mechanical framework is needed for low-temperature systems. At low temperatures one usually expects, due to the attractive forces between the atoms, to find the system in a crystalline solid phase in which the translational symmetry is broken and there is long-range-order. However, in a very dilute quantum system in which the short-range cohesive forces are not sufficiently strong, another kind of order appears, denoted as off-diagonal long-range-order [40]. When the De Broglie thermal wavelength  $\Lambda$  of the bosonic atoms is much larger than the mean interparticle spacing  $d$ , the superposition between atoms results in a macroscopic occupation of the zero-momentum single particle state. This means that, in the thermodynamic limit, a finite fraction of the particles occupy the lowest single particle energy state, namely the system forms a Bose-Einstein condensate. Notice that the occupation of the zero-momentum state implies that the atoms in the condensate are completely delocalized throughout the whole system.

We denote the number density of the atoms in the condensate with  $n_0$  and the fraction of particles in the condensate with  $n_0/n$ . The condensate fraction  $n_0/n$  of a system depends on several factors: on the presence/absence of interaction between bosons, on the specific form of the interaction, on the temperature and on the spatial dimension  $D$ . In this section we will derive general equations which take into account these conditions. In particular, one of the goals of this thesis is to calculate the condensate fraction  $n_0/n$  for interacting systems of bosonic cold atoms at a finite temperature  $T$ .

As an introduction to the general results, we calculate the condensate fraction  $n_0/n$  of a non-interacting three-dimensional Bose gas, which is one of the most famous results in the field of Bose-Einstein condensation. To reproduce it, we start from the grand potential per unit of volume  $\Omega/L^D$  of Eq. (1.61) in which we put  $D = 3$  and we take the interaction  $V(\vec{k}) = 0$ , obtaining

$$\frac{\Omega}{L^3} = -\mu\psi_0^2 + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \chi_k + \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\beta\chi_k}) \quad (2.13)$$

where we have defined the shifted free-particle spectrum  $\chi_k$  as

$$\chi_k = \frac{\hbar^2 k^2}{2m} - \mu \quad (2.14)$$

Notice that the second term of Eq. (2.13) can be regularized thank to the Veltman conjecture (1.73) and does not contribute to the grand potential. Similarly to what has been done in the previous section, the value of the order parameter  $\psi_0$  can be fixed by imposing the saddle point condition

$$\left( \frac{\partial \Omega_0(\mu, \psi_0)}{\partial \psi_0} \right)_{\mu, T, L^D} = 0 \quad (2.15)$$

from which we get the saddle point condition for a non-interacting system

$$\psi_0 = \begin{cases} 0 & \text{if } \mu < 0 \\ \text{any value,} & \text{if } \mu = 0 \end{cases} \quad (2.16)$$

The saddle point condition shows that, in presence of spontaneous symmetry breaking of the U(1) global symmetry, the order parameter  $\psi_0$  can assume a non-zero mean field value if the chemical potential is  $\mu = 0$ . Therefore we assume that, in a non-interacting system, there can be a finite condensate fraction in the thermodynamic limit if  $\mu = 0$  and we identify the condensate density with  $n_0 = \psi_0^2$ , i.e. as the square of the order parameter. Knowing the grand potential per unit of volume (2.13), we calculate the number density as

$$n = -\frac{1}{L^3} \frac{\partial \Omega}{\partial \mu} = n_0 + \frac{1}{\beta} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{e^{\beta \frac{\hbar^2 k^2}{2m}} - 1} \quad (2.17)$$

where we calculate the derivative of the grand potential per unit of volume with respect to the chemical potential  $\mu$  and, after that, we substitute the chemical potential value in the condensed phase  $\mu = 0$ . Notice that this procedure is actually different from the calculation of the number density described in the previous section after Eq. (2.9) and consider that the substitution of the saddle point condition (2.16) in the grand potential (2.13) *before* taking the derivative would not have given the same result of Eq. (2.17).

The critical temperature  $T_c$  for a three-dimensional system of noninteracting bosons can be derived by putting  $n_0 = 0$  in the Eq. (2.17) and performing the integral, obtaining

$$T_c = \frac{2\pi\hbar^2}{mk_B} \left( \frac{n}{\zeta(3/2)} \right)^{2/3} \quad (2.18)$$

where  $\zeta(3/2) \approx 2.612$  is the Riemann zeta function. The integration of the number density equation (2.17) for a generic temperature  $T$  allows us to obtain the dependence of condensate fraction  $n_0/n$  on the temperature

$$\frac{n_0}{n} = 1 - \left( \frac{T}{T_c} \right)^{3/2} \quad (2.19)$$

It should be noted that the condensate fraction of a three-dimensional system of noninteracting bosons goes from  $n_0/n = 1$  at  $T = 0$ , to  $n_0/n = 0$  at the critical temperature  $T_c$ , as displayed in Fig. (2.1).

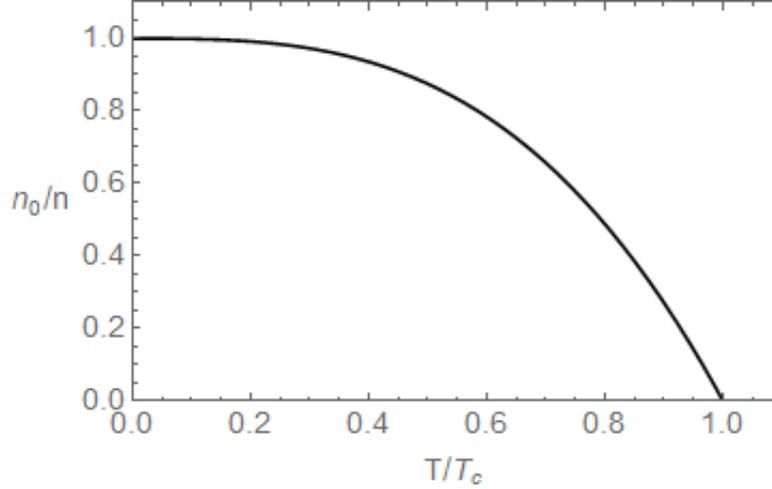


Figure 2.1: Condensate fraction  $n_0/n$  for noninteracting bosons in  $D = 3$ , reported as a function of the scaled temperature  $T/T_c$ . The critical temperature  $T_c$  is reported in Eq. (2.18)

The picture gets more complicated if, in the study of three-dimensional bosons, one includes the interaction. In this case the condensate fraction is lower than 1 even at  $T = 0$ . This phenomenon is usually denoted as quantum depletion and is linked to the fact that the interaction between particles depletes the condensate. Depending on the interaction strength, the zero-temperature quantum depletion may be more or less pronounced. While for dilute alkali cold atoms the condensate fraction  $n_0/n$  is approximately 1, He-II is a strongly-interacting bosonic system and less than 10% of the atoms are condensated at  $T = 0$  [36].

Bose-Einstein condensation can also be studied in low-dimensional systems. In this case, one must take into account that the Mermin-Wagner theorem demonstrates that [35], if the lagrangian is invariant under a continuous symmetry group ( $U(1)$  in our case), there cannot be long-range-order at finite temperature in two- and one- dimensional systems. In the thesis we will discuss only two-dimensional systems, in which the Mermin-Wagner theorem implies that there can be Bose-Einstein condensation only at  $T = 0$ , while there cannot be long-range-order (then the condensate phase) at  $T > 0$ , since at finite temperature the  $U(1)$  symmetry cannot be broken. In two-dimensional systems, one usually refers to the quasi-condensate, which can be regarded as a local condensate or, equivalently, the condensation in a system which is not considered is the thermodynamic limit, in which the condensate fraction must be zero.

Let us derive a zero-temperature formula for the number density  $n$  of a system in  $D$  spatial dimensions. We consider bosons interacting with a generic potential  $V(\vec{k})$ : the quantum fluctuations of the system, responsible of the quantum depletion, are taken

into account as the noninteracting quasi-particle excitations with spectrum  $E_{\vec{k}}$  of the zero-temperature gaussian grand potential (1.67). This density equation is an implicit expression which will be used to determine the condensate fraction  $n_0/n$  of the system. Denoting the zero-temperature grand potential with  $\Omega(\mu, \psi_0, T = 0)$ , constituted only by the contributions of Eqs. (1.62) and (1.67), we calculate the system density  $n$  at  $T = 0$  as [45]

$$n(n_0, T = 0) = -\frac{1}{L^D} \left. \frac{\partial \Omega(\mu, \psi_0, T = 0)}{\partial \mu} \right|_{\substack{\mu = g_0 \psi_0^2 \\ \psi_0^2 = n_0}} \quad (2.20)$$

where we take the derivative of the grand potential with respect to the chemical potential  $\mu$  first and then we substitute the mean field value of the chemical potential  $\mu = g_0 \psi_0^2$  (see the saddle point condition of Eq. (2.2)), with the identification for the condensate density  $n_0 = \psi_0^2$ . We justify this procedure considering that the number density  $n$  expressed in terms of the condensate density  $n_0$  has been calculated in the same way in the Eq. (2.17) of the three-dimensional noninteracting case.

Inserting the zero-temperature grand potential in Eq. (2.20), one obtains the density as a sum of two contributions

$$n(n_0, T = 0) = n_0 + f_g^{(0)}(n_0) \quad (2.21)$$

where we have defined the gaussian contribution to the zero-temperature density  $f_g^{(0)}(n_0)$  as

$$f_g^{(0)}(n_0) = -\frac{1}{2L^D} \sum_{\vec{k}} \left. \frac{\partial E_{\vec{k}}}{\partial \mu}(\mu, \psi_0) \right|_{\substack{\mu = g_0 \psi_0^2 \\ \psi_0^2 = n_0}} \quad (2.22)$$

To calculate it, we notice that the derivative with respect to  $\mu$  of the general excitation spectrum  $E_{\vec{k}}$  of Eq. (1.65) is given by

$$\left. \frac{\partial E_{\vec{k}}}{\partial \mu}(\mu, \psi_0) \right|_{\substack{\mu = g_0 \psi_0^2 \\ \psi_0^2 = n_0}} = -\frac{\frac{\hbar^2 k^2}{2m} + \frac{n_0}{2}(V(\vec{k}) + V(-\vec{k}))}{\sqrt{(\frac{\hbar^2 k^2}{2m} + \frac{n_0}{2}(V(\vec{k}) + V(-\vec{k})))^2 - (n_0 V(\vec{k}))^2}} \quad (2.23)$$

Substituting this result in the zero-temperature gaussian contribution to the density (2.22), we can write  $f_g^{(0)}(n_0)$  in the thermodynamic limit as

$$f_g^{(0)}(n_0) = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \frac{\frac{\hbar^2 k^2}{2m} + \frac{n_0}{2}(V(\vec{k}) + V(-\vec{k}))}{\sqrt{(\frac{\hbar^2 k^2}{2m} + \frac{n_0}{2}(V(\vec{k}) + V(-\vec{k})))^2 - (n_0 V(\vec{k}))^2}} \quad (2.24)$$

which depends on the interaction  $V(\vec{k})$  and will be calculated explicitly choosing a zero and a finite-range interaction in the next chapter. The relation (2.21) can then be used to obtain a relation between  $n$  and  $n_0$  at zero-temperature, and in principle, also the condensate fraction  $n_0/n$  and the quantum depletion  $\frac{n-n_0}{n}$ .

Consider that, in the equations above, we express the zero-temperature density  $n(n_0, T = 0)$  as a function of the condensate density  $n_0$ . At finite temperature, as we have previously stated, the spatial dimension  $D$  of the system plays a crucial role in the phenomenon of Bose-Einstein condensation. In three-dimensional systems, the condensate fraction  $n_0/n$  is also reduced by the effect of thermal fluctuations, encoded in the term  $\Omega_g^{(T)}$ . Quite differently, for low-dimensional systems it is expected that the thermal fluctuations diverge, destroying the zero-temperature condensate. Keeping this distinction in mind, we continue to use a generic spatial dimension  $D$  and we express the number density  $n$  extending the relation (2.21) at finite temperature as

$$n(n_0, T) = n_0 + f_g^{(0)}(n_0) + f_g^{(T)}(n_0) \quad (2.25)$$

where  $f_g^{(0)}(n_0)$  is given as before by Eq. (2.24), and where we have defined the finite-temperature density contribution  $f_g^{(T)}(n_0)$  as

$$f_g^{(T)}(n_0) = -\frac{1}{L^D} \left. \frac{\partial \Omega_g^{(T)}(\mu, \psi_0)}{\partial \mu} \right|_{\substack{\mu = g_0 \psi_0^2 \\ \psi_0^2 = n_0}} = -\int \frac{d^D k}{(2\pi)^D} \frac{1}{e^{\beta E_{\vec{k}}} - 1} \left. \frac{\partial E_{\vec{k}}}{\partial \mu}(\mu, \psi_0) \right|_{\substack{\mu = g_0 \psi_0^2 \\ \psi_0^2 = n_0}} \quad (2.26)$$

If we substitute in the third member the explicit form of the excitation spectrum derivative (2.23), we get

$$f_g^{(T)}(n_0) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{e^{\beta E_{\vec{k}}} - 1} \frac{\frac{\hbar^2 k^2}{2m} + \frac{n_0}{2}(V(\vec{k}) + V(-\vec{k}))}{\sqrt{(\frac{\hbar^2 k^2}{2m} + \frac{n_0}{2}(V(\vec{k}) + V(-\vec{k})))^2 - (n_0 V(\vec{k}))^2}} \quad (2.27)$$

We maintain the generic dimension  $D$  for regularization necessities, even though we will show in the next chapter that  $f_g^{(T)}(n_0)$  diverges for low-dimensional systems, due to the Mermin-Wagner theorem.

Finally, as a consistence test of this procedure, we expect to obtain the critical condensation temperature  $T_c$  of Eq. (2.18) if we choose  $D = 3$ , put the condensate density  $n_0$  to zero and we choose  $V(\vec{k}) = 0$  in the number density (2.25). This is actually correct, since the density equation (2.25) becomes

$$n(n_0 = 0, T_c) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{e^{\frac{1}{k_B T_c} \frac{\hbar^2 k^2}{2m}} - 1} \quad (2.28)$$

We simply delete the first integral using the Veltman conjecture (1.73), while the second integral can be easily computed and used to obtain the number density at the critical temperature  $n(n_0 = 0, T_c)$ . It is straightforward to see that the inversion of this result produces exactly the critical temperature  $T_c$  of the Eq. (2.18).

In the next section we retrace the historical Lev Landau derivation of the normal density  $n_n$  of a superfluid, obtained in the context of the two-fluid model.

## 2.3 Landau derivation of the superfluid density

In 1941 Lev Landau proposed [28] a two-fluid model to explain the superfluid behavior of  $^4\text{He}$ . We now reproduce his derivation, as presented by himself in the Course of Theoretical Physics [29].

Let us consider a bosonic fluid whose excitation spectrum is given by  $E_p$ .<sup>1</sup> If the low energy part excitation spectrum is phononic (or gapped), thus linear (or constant) in low momenta  $\vec{p}$ , the fluid shows a property called superfluidity. A superfluid can flow through capillaries without showing a dissipation of its kinetic energy, namely as a fluid with zero viscosity.

To explain this phenomenon let us now consider a generic fluid flowing in a capillary with a velocity  $\vec{v}$ . We identify with  $\mathcal{S}$  this frame of reference, the one in which the capillary walls are stationary and the fluid is seen to move with velocity  $\vec{v}$ . Conversely we denote with  $\mathcal{S}'$  the frame of reference in which the capillary moves with velocity  $-\vec{v}$  and the fluid is still.

Since we are studying a low energy system we suppose that its physical description is invariant under the set of Galilean transformations

$$\begin{aligned}\vec{r} &= \vec{r}' + \vec{v}t \\ \vec{P} &= \vec{P}' + M\vec{v} \\ E &= E' + \frac{1}{2}Mv^2 + \vec{P}' \cdot \vec{v}\end{aligned}\tag{2.29}$$

where  $\vec{r}$  denotes the macroscopic position of a fluid element with mass  $M$ , momentum  $\vec{P}$  and energy  $E$  in the frame of reference  $\mathcal{S}$ , while the  $'$  mark is related to the same quantities in frame of reference  $\mathcal{S}'$ .

One can then write the energy and the momentum of the fluid in  $\mathcal{S}'$  and calculate the same quantities in  $\mathcal{S}$  employing the transformations (2.29). For a fluid initially at rest in  $\mathcal{S}'$  we have

$$\begin{aligned}\mathcal{S}' : \quad E' &= \epsilon_0 \quad \vec{P}' = 0 \\ \mathcal{S} : \quad E &= \epsilon_0 + \frac{1}{2}Mv^2 \quad \vec{P} = M\vec{v}\end{aligned}\tag{2.30}$$

where  $\epsilon_0$  is the macroscopic energy of the fluid in  $\mathcal{S}'$ , where it is at rest. If the system is in the ground state, then simply  $\epsilon_0 = 0$ .

Let us now analyze how the energy changes during the motion of the fluid.

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<sup>1</sup>In this section  $\vec{p}$  represents the momentum of the excitation with spectrum  $E_p$ , while  $\vec{k}$  that appears in the excitation spectrum (1.65) is a wavevector. Since we are not choosing  $\hbar = 1$  they are not the same quantity and the relations obtained can be written in terms of the wavevector  $\vec{k}$  by putting  $\vec{p} = \hbar\vec{k}$ . Here we choose this convention for uniformity with the work of Landau.

A viscous fluid in  $\mathcal{S}$  will perceive the friction against the capillary wall and the friction within the fluid itself. As a result of this process, the fluid will gradually decrease its macroscopic kinetic energy. Conversely, in  $\mathcal{S}'$ , the fluid will acquire kinetic energy due to the friction between the moving capillary wall and the fluid itself. In this case, the fluid is not put in motion as a whole, but it is gradually excited by the walls: let us suppose that, because of the friction, an elementary excitation with momentum  $-\vec{p}$  and spectrum  $E_p$  appears. Then, we rewrite the previous equations as

$$\begin{aligned} \mathcal{S}' : \quad E' &= \epsilon_0 + E_p & \vec{P}' &= -\vec{p} \\ \mathcal{S} : \quad E &= \epsilon_0 + E_p + \frac{1}{2}Mv^2 - \vec{p} \cdot \vec{v} & \vec{P} &= -\vec{p} + M\vec{v} \end{aligned} \quad (2.31)$$

We see that, due to the friction, the energy of the fluid in system  $\mathcal{S}$  changes of  $\Delta E$ , where

$$\Delta E = E_p - \vec{p} \cdot \vec{v} \quad (2.32)$$

For a normal viscous fluid this quantity must be negative, which allows us to identify the minimum velocity it must have in order to dissipate its energy

$$v_{\min} = \lim_{p \rightarrow 0} \frac{E_p}{p} \quad (2.33)$$

If  $v_{\min} = 0$  the fluid is ordinary and it flows dissipating its energy, but if  $v_{\min}$  is finite and positive the fluid behaves like a superfluid for velocities  $v < v_{\min}$ . Therefore, a fluid whose excitation spectrum is either phononic or gapped possess a finite minimum velocity  $v_{\min} = \lim_{p \rightarrow 0} \frac{E_p}{p} \neq 0$ , under which it shows a superfluid behavior. This result is usually known as the Landau criterion for superfluidity.

In the previous equations we have considered a superfluid in the ground state at  $T = 0$ . Let us now consider a fluid system at a low, but non-zero temperature. In this case the fluid will not be in the ground state, but in an excited state due to the presence of thermal excitations. As we have seen in the previous argument, the presence of excitations is responsible for the viscosity of the fluid, i.e. for the dissipation of its macroscopic kinetic energy. Therefore, we may think the gas of non-interacting quasi-particles as the normal, viscous part of the fluid, and we will suppose that it moves with velocity  $\vec{v}$  relative to the remaining superfluid liquid.

We are interested in calculating the momentum density of the normal part as a thermal mean over all excitations: to perform the calculation we will need to know the distribution function of the quasi-particles. The main problem is that the distribution function of the gas must be obtained in a frame of reference at rest with the gas. For this scope, we now identify the frames of reference  $\mathcal{S}$  where the excitations gas is at rest and the fluid moves with velocity  $-\vec{v}$  and  $\mathcal{S}'$ , where the excitations move with velocity  $\vec{v}$  and the fluid is at rest.

The previous argument can be used again, obtaining exactly the Eqs. (2.30) where  $\epsilon_0$  can be now interpreted as the energy of the system in  $\mathcal{S}'$ . After the appearance of

an excitation in  $\mathcal{S}'$  one obtains again the equations (2.31), which allow us to identify the energy of a single excitation in frame of reference  $\mathcal{S}'$  in which the gas is at rest as  $E_p - \vec{p} \cdot \vec{v}$ . This means that the bosonic non-interacting gas of quasi-particles will follow the "shifted" Bose-Einstein distribution function  $f_B(E_p - \vec{p} \cdot \vec{v})$ , with

$$f_B(E_p - \vec{p} \cdot \vec{v}) = \frac{1}{e^{\beta(E_p - \vec{p} \cdot \vec{v})} - 1} \quad (2.34)$$

Knowing the particle distribution in the frame of reference  $\mathcal{S}'$ , we calculate the total momentum per unit of volume  $\vec{\mathcal{P}}_n$  of the quasi-particle excitations gas as

$$\vec{\mathcal{P}}_n = \frac{1}{L^D} \sum_{\vec{p}} \vec{p} f_B(E_p - \vec{p} \cdot \vec{v}) \quad (2.35)$$

which can be easily written in the thermodynamic limit as

$$\vec{\mathcal{P}}_n = \int \frac{d^D p}{(2\pi\hbar)^D} \vec{p} f_B(E_p - \vec{p} \cdot \vec{v}) \quad (2.36)$$

According to the Landau criterion, there can be superfluid motion only if the modulus of the velocity is  $v < v_{\min}$ . Therefore we consider a fluid moving with a small velocity  $\vec{v}$ , such as we can expand the distribution function (2.34) around  $\vec{v} = 0$ , obtaining

$$\vec{\mathcal{P}}_n = - \int \frac{d^D p}{(2\pi\hbar)^D} \vec{p} (\vec{p} \cdot \vec{v}) \frac{\partial f_B(E_p)}{\partial E_p} \quad (2.37)$$

where we report only the first order of the expansion of  $f_B$ , since the zero-order term is zero due to the symmetry of the integration domain. Finally, we may simplify Eq. (2.37) by taking the average over the directions of  $\vec{p}$ , according to the property  $\int d^D p p_i p_j v^j F(|\vec{p}|) = \frac{1}{D} \delta_{ij} v^j \int d^D p p^2 F(|\vec{p}|)$ , where  $F(|\vec{p}|)$  is a generic function which depends only on the modulus of the momentum  $\vec{p}$ . The momentum density  $\vec{\mathcal{P}}_n$  reads

$$\vec{\mathcal{P}}_n = \frac{\beta}{D} \vec{v} \int \frac{d^D p}{(2\pi\hbar)^D} p^2 \frac{e^{\beta E_p}}{(e^{\beta E_p} - 1)^2} \quad (2.38)$$

At last, we calculate the normal number density  $n_n$  considering that the momentum density  $\vec{\mathcal{P}}_n$  is the momentum per unit of volume transported by the normal part of the fluid:  $\vec{\mathcal{P}}_n = n_n m \vec{v}$ . We can identify<sup>2</sup>

$$n_n = \frac{\beta}{Dm} \int \frac{d^D k}{(2\pi)^D} \hbar^2 k^2 \frac{e^{\beta E_k}}{(e^{\beta E_k} - 1)^2} \quad (2.39)$$

which is the normal density of the fluid, as originally obtained by Landau. We are in this way implementing a two-fluid model: we are studying the superfluid as a mixture

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<sup>2</sup>For uniformity of notation with the rest of the thesis, we restart using the wavevector  $\vec{k}$ , i.e. we put  $\vec{p} = \hbar \vec{k}$

of the normal component  $n_n$  and a superfluid component  $n_s$ , which constitute together the fluid with density  $n$ , such as

$$n = n_s + n_n \quad (2.40)$$

Despite the two-fluid model is very useful to understand the physical behavior of a fluid at low temperature, it should be noted that the two components are not physically separable. Instead, we should view the fluid as simultaneously capable of two different motions: the superfluid non-viscous flow interesting a fraction  $n_s/n$  of the fluid and the normal viscous flow interesting a fraction  $n_n/n$  of the fluid. Depending on the temperature, the fluid shows a different behavior since a different proportion of the two motions appear: the fluid density goes from  $n = n_s$  at  $T = 0$  to  $n = n_n$  at the transition temperature.

In the next section we will follow a microscopic approach to derive an expression of the normal density  $n_n$  in agreement with the familiar Landau result of Eq. (2.39). Moreover, we will use the finite-temperature relationship between the total number density  $n$  of the system and the condensate density  $n_0$  of Eq. (2.25) to derive an explicit equation connecting in three dimensions the superfluid density  $n_s$  with the condensate density  $n_0$  and the temperature  $T$ , which may be regarded as an explicit formulation of the Josephson relation [21].

## 2.4 Microscopic derivation of the superfluid density

We introduce here a derivation to calculate the normal fluid density  $n_n$  from a microscopic approach, obtaining an equivalent result to the Landau expression (2.39). The normal density, expressed as a function of the condensate density  $n_0$  and the temperature  $T$ , will be used, together with the density equation (2.25), to derive the superfluid density  $n_s$ .

We start from the non-relativistic lagrangian (1.36) of the bosonic system, rewritten in the equivalent form

$$\mathcal{L}(\bar{\psi}, \psi) = \bar{\psi}(\vec{r}, \tau) \left( \hbar \partial_\tau - \frac{\hbar^2 \nabla^2}{2m} - \mu \right) \psi(\vec{r}, \tau) + \frac{1}{2} \int d^D r' |\psi(\vec{r}, \tau)|^2 V(\vec{r}, \vec{r}') |\psi(\vec{r}', \tau)|^2 \quad (2.41)$$

We consider a fluid moving with velocity  $\vec{v}$  with respect to the laboratory frame of reference. Like in the study of fluid mechanics [27], the description of a fluid in motion can be made substituting the partial time derivative with the Lagrangian fluid velocity, following the motion of the fluid

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \vec{v} \cdot \vec{\nabla} \quad (2.42)$$

However, remembering that we are working with imaginary time, we need to perform the Wick rotation  $\tau = it$ , obtaining the relation

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau'} - i\vec{v} \cdot \vec{\nabla} \quad (2.43)$$

Moreover, we treat the bosonic fluid in the two-fluid model framework, thus as composed by a normal part and a superfluid part. Since the superfluid part of the fluid flows without internal friction and without exchanging momentum with the normal part of the fluid, the velocity  $\vec{v}_s$  may be, in principle, different from  $\vec{v}$ . Therefore, we include a superfluid current in the lagrangian description rewriting the bosonic field with the phase twist [16, 20]

$$\psi'(\vec{r}, \tau) = e^{i \frac{m \vec{v}_s \cdot \vec{r}}{\hbar}} \psi(\vec{r}, \tau) \quad (2.44)$$

Considering the Lagrangian time derivative of Eq. (2.42) and the field transformation given by Eq. (2.44), we rewrite the lagrangian (2.41) in terms of the new field  $\psi'(\vec{r}, \tau)$  as

$$\begin{aligned} \mathcal{L}(\bar{\psi}', \psi') = \bar{\psi}'(\vec{r}, \tau) \left( \hbar \partial_\tau - \frac{\hbar^2 \nabla^2}{2m} - \mu + \frac{1}{2} m \vec{v}_s \cdot (\vec{v}_s - 2\vec{v}) + (\vec{v} - \vec{v}_s) \cdot (-i\hbar \vec{\nabla}) \right) \psi'(\vec{r}, \tau) + \\ \frac{1}{2} \int d^D r' |\psi'(\vec{r}, \tau)|^2 V(\vec{r}, \vec{r}') |\psi'(\vec{r}', \tau)|^2 \end{aligned} \quad (2.45)$$

Notice that the interacting part of the new lagrangian does not change at all with respect to the initial lagrangian (2.41), due to the translational invariance of the spatial integration (both for the  $\vec{r}'$  coordinate and for  $\vec{r}$ ) since in the functional integral approach the lagrangian is always thought to be inside the action (1.35). In the new lagrangian appeared the term  $(\vec{v} - \vec{v}_s) \cdot (-i\hbar\vec{\nabla})$ , which is obtained as a consequence of the Lagrangian time derivative and the phase twist. However, among the new terms, the most interesting is the fourth term of the new lagrangian (2.41), since it produces a shift of the chemical  $\mu$  to the effective value  $\mu_{\text{eff}}$  given by

$$\mu_{\text{eff}} = \mu - \frac{1}{2}m\vec{v}_s \cdot (\vec{v}_s - 2\vec{v}) \quad (2.46)$$

It is interesting to see that the existence of a superfluid current with velocity  $\vec{v}_s \neq 0$  in the system is responsible for a shift of the chemical potential. In fact, the phase of the bosonic field is proportional to the system chemical potential  $\mu$  [40], therefore, due the presence of a superfluid current, the phase twist (2.44) appears and changes the chemical potential to  $\mu_{\text{eff}}$ .

From now on we will drop the mark ' on the fields for easiness of notation, rewriting the lagrangian (2.45) as

$$\mathcal{L}(\bar{\psi}, \psi) = \bar{\psi}(\vec{r}, \tau) \left( \hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m} - \mu_{\text{eff}} + (\vec{v} - \vec{v}_s) \cdot (-i\hbar\vec{\nabla}) \right) \psi(\vec{r}, \tau) + \frac{1}{2} \int d^D r' |\psi(\vec{r}, \tau)|^2 V(\vec{r}, \vec{r}') |\psi(\vec{r}', \tau)|^2 \quad (2.47)$$

Starting from this lagrangian, all the operations described in the section (1.3) can be repeated from the broken-symmetry field parametrization (1.43) to the action (1.53). Since the extension of the previous arguments is straightforward, we refer the reader to the section (1.3) for the details. In this context we obtain the system action  $S$  in the Fourier space as a function of the complex fluctuation field  $\eta(k)$  around the homogeneous order parameter  $\psi_0$  written in the following form

$$S[\eta(k), \eta(-k)] = S_0 + \frac{\hbar}{2} \sum_{\vec{k}\omega_n} (\bar{\eta}(k) \quad \eta(-k)) \mathbf{M}(k) \begin{pmatrix} \eta(k) \\ \bar{\eta}(-k) \end{pmatrix} \quad (2.48)$$

where we define the  $2 \times 2$  matrix  $\mathbf{M}(k)$  as

$$\mathbf{M}(k) = \beta \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \quad (2.49)$$

with

$$\begin{aligned} \mathbf{M}_{11} &= -i\hbar\omega_n + \frac{\hbar^2 k^2}{2m} - \mu_{\text{eff}} + g_0\psi_0^2 + \frac{\psi_0^2}{2}(V(\vec{k}) + V(-\vec{k})) + \hbar(\vec{v} - \vec{v}_s) \cdot \vec{k} \\ \mathbf{M}_{22} &= +i\hbar\omega_n + \frac{\hbar^2 k^2}{2m} - \mu_{\text{eff}} + g_0\psi_0^2 + \frac{\psi_0^2}{2}(V(\vec{k}) + V(-\vec{k})) - \hbar(\vec{v} - \vec{v}_s) \cdot \vec{k} \\ \mathbf{M}_{12} &= \mathbf{M}_{21} = \psi_0^2 V(\vec{k}) \end{aligned} \quad (2.50)$$

As before, we obtain the grand potential as the sum of two contributions

$$\Omega = \Omega_0 + \Omega_g \quad (2.51)$$

where the order parameter grand potential  $\Omega_0$  now contains  $\mu_{\text{eff}}$  instead of  $\mu$  as in Eq. (1.62)

$$\Omega_0 = L^D \left( -\mu_{\text{eff}} \psi_0^2 + \frac{1}{2} g_0 \psi_0^4 \right) \quad (2.52)$$

and the gaussian grand potential  $\Omega_g$  is given by

$$\Omega_g = -\frac{1}{\beta} \sum_{\substack{\vec{k} \omega_n \\ k_z > 0}} \ln[\det \mathbf{M}(k)]^{-1} = \frac{1}{2\beta} \sum_{\vec{k} \omega_n} \ln[\beta^2 (E_{\vec{k}}^2 - (i\hbar\omega_n - \hbar(\vec{v} - \vec{v}_s) \cdot \vec{k})^2)] \quad (2.53)$$

The excitation spectrum of the system  $E_{\vec{k}}$  is simply given by the previous excitation spectrum of Eq. (1.65), in which the chemical potential  $\mu$  is replaced by  $\mu_{\text{eff}}$ , namely

$$E_{\vec{k}} = \sqrt{\left( \frac{\hbar^2 k^2}{2m} - \mu_{\text{eff}} + g_0 \psi_0^2 + \frac{\psi_0^2}{2} (V(\vec{k}) + V(-\vec{k})) \right)^2 - (\psi_0^2 V(\vec{k}))^2} \quad (2.54)$$

However, differently from the previous chapter, the summation over the Matsubara frequencies in the gaussian grand potential (2.53) cannot be directly made due to the presence of the linear term in the wavevector  $\vec{k}$ . Let us employ the logarithm properties to rewrite  $\Omega_g$  in the equivalent form

$$\begin{aligned} \Omega_g = & \frac{1}{2\beta} \sum_{\vec{k} \omega_n} \ln[\beta(E_{\vec{k}} - (i\hbar\omega_n - \hbar(\vec{v} - \vec{v}_s) \cdot \vec{k}))] + \\ & \frac{1}{2\beta} \sum_{\vec{k} \omega_n} \ln[\beta(E_{\vec{k}} + (i\hbar\omega_n - \hbar(\vec{v} - \vec{v}_s) \cdot \vec{k}))] \end{aligned} \quad (2.55)$$

In this case, we cannot use the procedure described in the Appendix A, unless we are not able to write the gaussian grand potential in the form given by Eq. (A.2). Hence, to perform the Matsubara frequencies summation we need to make some simplifying assumptions. First of all, we notice that under the hypothesis that the interaction potential is left unchanged by a reflection of the wavevector

$$V(\vec{k}) = V(-\vec{k}) \quad (2.56)$$

it follows that the also excitation spectrum (2.54) is symmetric under a reflection of the wavevector  $E_{\vec{k}} = E_{-\vec{k}}$ . We now swap  $\vec{k}$  with  $-\vec{k}$  in the summation over the wavevectors  $\vec{k}$  in the second term of Eq. (2.55) and we define the shifted excitation spectrum  $\xi_{\vec{k}}$  as

$$\xi_{\vec{k}} = E_{\vec{k}} + \hbar(\vec{v} - \vec{v}_s) \cdot \vec{k} \quad (2.57)$$

The logarithm properties, used to factor the gaussian grand potential  $\Omega_g$ , can be used again to rewrite it in the form

$$\Omega_g = \frac{1}{2\beta} \sum_{\vec{k}} \sum_{n=-\infty}^{+\infty} \ln[\beta^2(\hbar^2\omega_n^2 + \xi_{\vec{k}}^2)] \quad (2.58)$$

We can now follow the procedure described in Appendix A to sum over the Matsubara frequencies  $\omega_n$ . Finally, we obtain the total grand potential  $\Omega$  in the familiar form

$$\Omega = \Omega_0 + \Omega_g^{(0)} + \Omega_g^{(T)} \quad (2.59)$$

with  $\Omega_0$  given by Eq. (2.52), while the zero-temperature grand potential is

$$\Omega_g^{(0)} = \frac{1}{2} \sum_{\vec{k}} \xi_{\vec{k}} = \frac{1}{2} \sum_{\vec{k}} E_{\vec{k}} \quad (2.60)$$

where we should have also added the term  $\sum_{\vec{k}} \hbar(\vec{v} - \vec{v}_s) \cdot \vec{k}$ , which however is zero for symmetry reasons and where the excitation spectrum  $E_{\vec{k}}$ , due to the assumption (2.56) on the interaction potential, becomes

$$E_{\vec{k}} = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu_{\text{eff}} + \psi_0^2(g_0 + V(\vec{k}))\right)^2 - (\psi_0^2 V(\vec{k}))^2} \quad (2.61)$$

The finite-temperature grand potential  $\Omega_g^{(T)}$  now reads

$$\Omega_g^{(T)} = \frac{1}{\beta} \sum_{\vec{k}} \ln\left(1 - e^{-\beta(E_{\vec{k}} + \hbar(\vec{v} - \vec{v}_s) \cdot \vec{k})}\right) \quad (2.62)$$

Let us remark that in the derivation we have made the crucial assumption (2.56) that the interaction potential is symmetric under the reflection of the wavevector  $\vec{k}$ . Considering that the interaction  $V(\vec{r} - \vec{r}')$  can be written in terms of the Fourier component  $V(\vec{k})$  as the Fourier transform (1.39), one can show that the hypothesis of invariance under reflection in the Fourier space implies the same invariance in real space, namely

$$V(\vec{r} - \vec{r}') = V(\vec{r}' - \vec{r}) \quad (2.63)$$

The knowledge of the grand potential  $\Omega$  in the form (2.59) constitutes the starting point to determine the superfluid number density  $n_s$  of the fluid. According to the two-fluid model of a superfluid, we expect to obtain  $n_s$  as

$$n_s = n - n_n \quad (2.64)$$

In section (2.2) we have obtained the finite-temperature number density  $n(n_0, T)$  in Eq. (2.25). Here we calculate also the normal density  $n_n(n_0, T)$ , again as a function of the

condensate density  $n_0$  and the temperature  $T$ : this will allow us to obtain an explicit formulation of the Josephson relation for  $n_s$ .

We start by considering that, since the chemical potential  $\mu$  is shifted to the effective value  $\mu_{\text{eff}}$  given by Eq. (2.46) due to the presence of the superfluid current, the saddle point condition (2.2) now becomes

$$\psi_0 = \sqrt{\frac{\mu_{\text{eff}}}{g_0}} \quad (2.65)$$

Notice that all the contributions calculated in the equation for the finite-temperature number density  $n(n_0, T)$  obtained in section (2.2) continue to hold, provided that  $\mu_{\text{eff}} = g_0\psi_0^2 = g_0n_0$ . We remark this point because the density contributions of Eq. (2.25) will emerge again in the following equations.

Let us calculate the total momentum density  $\vec{\mathcal{P}}$  of the fluid as

$$\vec{\mathcal{P}} = \frac{1}{L^D} \frac{\partial \Omega(\mu_{\text{eff}}, \psi_0)}{\partial(-\vec{v})} \Bigg|_{\substack{\mu_{\text{eff}} = g_0\psi_0^2 \\ \psi_0^2 = n_0}} \quad (2.66)$$

where, in analogy with what has been done for the system density, we calculate the derivative of the grand potential  $\Omega$  with respect to the opposite of the fluid velocity  $-\vec{v}$  and after that we substitute the condition  $\mu_{\text{eff}} = g_0\psi_0^2 = g_0n_0$ . The total momentum density  $\vec{\mathcal{P}}$  is given by

$$\vec{\mathcal{P}} = \vec{\mathcal{P}}_0 + \vec{\mathcal{P}}_g^{(0)} + \vec{\mathcal{P}}_g^{(T)} \quad (2.67)$$

where, since the grand potential  $\Omega$  is constituted by the three contributions of Eq. (2.59), we define  $\vec{\mathcal{P}}_0$ ,  $\vec{\mathcal{P}}_g^{(0)}$  and  $\vec{\mathcal{P}}_g^{(T)}$  accordingly: let us calculate separately each one of them.

The homogeneous momentum density  $\vec{\mathcal{P}}_0$  is given by

$$\vec{\mathcal{P}}_0 = n_0 m \vec{v}_s \quad (2.68)$$

The zero-temperature gaussian contribution  $\vec{\mathcal{P}}_g^{(0)}$  is

$$\vec{\mathcal{P}}_g^{(0)} = f_g^{(0)}(n_0) m \vec{v}_s \quad (2.69)$$

where  $f_g^{(0)}(n_0)$  is given by Eq. (2.22).

The finite-temperature gaussian contribution  $\vec{\mathcal{P}}_g^{(T)}$  to the momentum density needs more work, since it is more involved. We find

$$\vec{\mathcal{P}}_g^{(T)} = \int \frac{d^D k}{(2\pi)^D} \frac{1}{e^{\beta(E_{\vec{k}} + \hbar(\vec{v} - \vec{v}_s) \cdot \vec{k})} - 1} \left[ \frac{\partial E_{\vec{k}}}{\partial \mu_{\text{eff}}} \frac{\partial \mu_{\text{eff}}}{\partial(-\vec{v})} - \hbar \vec{k} \right] \Bigg|_{\substack{\mu_{\text{eff}} = g_0\psi_0^2 \\ \psi_0^2 = n_0}} \quad (2.70)$$

Assuming that the difference between the velocity  $\vec{v}$  of the normal fluid and the velocity  $\vec{v}_s$  of the superfluid is small, we can expand the exponential at the denominator of the integral for  $\vec{v} - \vec{v}_s \approx 0$ . Then, taking into account that the integrals of some of the terms in the expansion give zero for symmetry reasons, we obtain

$$\begin{aligned} \vec{\mathcal{P}}_g^{(T)} = & \int \frac{d^D k}{(2\pi)^D} \frac{1}{e^{\beta E_{\vec{k}}} - 1} \frac{\partial E_{\vec{k}}}{\partial \mu_{\text{eff}}} \frac{\partial \mu_{\text{eff}}}{\partial (-\vec{v})} \Bigg|_{\substack{\mu_{\text{eff}} = g_0 \psi_0^2 \\ \psi_0^2 = n_0}} + \\ & \beta \int \frac{d^D k}{(2\pi)^D} \frac{e^{\beta E_{\vec{k}}}}{(e^{\beta E_{\vec{k}}} - 1)^2} \hbar^2 ((\vec{v} - \vec{v}_s) \cdot \vec{k}) \vec{k} \Bigg|_{\substack{\mu_{\text{eff}} = g_0 \psi_0^2 \\ \psi_0^2 = n_0}} \end{aligned} \quad (2.71)$$

In the first term, after the calculation of the derivative of the effective chemical potential  $\mu_{\text{eff}}$ , we recognize the definition of the finite-temperature contribution  $f_g^{(T)}(n_0)$  of Eq. (2.27), while the second term is formally analogous to the momentum density of the normal fluid (2.37) obtained by Landau. In particular, we find that, using explicit vectorial components, the finite-temperature gaussian contribution  $\vec{\mathcal{P}}_g^{(T)}$  can be written as

$$(\mathcal{P}_g^{(T)})_i = f_g^{(T)}(n_0) m (\vec{v}_s)_i + (n_n)_{ij} m (\vec{v} - \vec{v}_s)_j \quad (2.72)$$

where we identify the tensorial normal density  $(n_n)_{ij}$  as

$$(n_n)_{ij} = \frac{\beta \hbar^2}{m} \int \frac{d^D k}{(2\pi)^D} \frac{e^{\beta E_{\vec{k}}}}{(e^{\beta E_{\vec{k}}} - 1)^2} k_i k_j \Bigg|_{\substack{\mu_{\text{eff}} = g_0 \psi_0^2 \\ \psi_0^2 = n_0}} \quad (2.73)$$

Here the normal density is obtained in general as a tensorial quantity, however, following the work of Landau, we focus on the description of a system in which the excitation spectrum  $E_{\vec{k}}$  depends only on the modulus  $|\vec{k}|$  of the wavevector. Hence, we can rewrite the tensorial normal density (2.73) using the isotropy property for a generic integrand function  $F(|\vec{k}|)$  of  $|\vec{k}|$

$$\int d^D k F(|\vec{k}|) k_i k_j = \frac{1}{D} \delta_{ij} \int d^D k k^2 F(|\vec{k}|) \quad (2.74)$$

such as the finite-temperature gaussian contribution  $\vec{\mathcal{P}}_g^{(T)}$  of Eq. (2.72) becomes

$$\vec{\mathcal{P}}_g^{(T)} = f_g^{(T)}(n_0) m \vec{v}_s + n_n(n_0, T) m (\vec{v} - \vec{v}_s) \quad (2.75)$$

where we have defined the scalar normal density  $n_n(n_0, T)$  as

$$n_n(n_0, T) = \frac{\beta}{mD} \int \frac{d^D k}{(2\pi)^D} \hbar^2 k^2 \frac{e^{\beta E_k(n_0)}}{(e^{\beta E_k(n_0)} - 1)^2} \quad (2.76)$$

Notice the result of Landau reproduced in our Eq. (2.39) coincides with the normal density obtained here, in which the excitation spectrum  $E_k(n_0)$  is now given by the

spectrum (2.61) where it is substituted the condition  $\psi_0^2 = n_0$ , namely

$$E_k(n_0) = \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2n_0 V(k) \right)} \quad (2.77)$$

We now put the Eqs. (2.68), (2.69) and (2.75) together, obtaining the total momentum density  $\vec{\mathcal{P}}$  as

$$\vec{\mathcal{P}} = [n_0 + f_g^{(0)}(n_0) + f_g^{(T)}(n_0)] m\vec{v}_s + n_n(n_0, T) m(\vec{v} - \vec{v}_s) \quad (2.78)$$

We recognize inside the square bracket the number density  $n(n_0, T)$  as obtained in Eq. (2.25), so that  $\vec{\mathcal{P}}$  now becomes

$$\vec{\mathcal{P}} = n(n_0, T) m\vec{v}_s + n_n(n_0, T) m(\vec{v} - \vec{v}_s) \quad (2.79)$$

Finally, we employ the usual superfluid density  $n_s$  definition of Eq. (2.64), which we reinterpret as

$$n_s = n(n_0, T) - n_n(n_0, T) \quad (2.80)$$

to get the total momentum fluid density  $\vec{\mathcal{P}}$  of the two-fluid model

$$\vec{\mathcal{P}} = n_s m\vec{v}_s + n_n m\vec{v} \quad (2.81)$$

In this way we have expressed, following a microscopic approach, the fluid total momentum density as the momentum carried by the normal fluid with density  $n_n$  plus the momentum density of the superfluid with density  $n_s$ .

The main result of this thesis is given by Eq. (2.80), which highlights a non-trivial relationship between the superfluid density  $n_s$ , the condensate density  $n_0$  and the temperature  $T$ . In particular, we can regard it as an explicit formulation, valid at a gaussian level, of the Josephson relation which links  $n_s$ ,  $n_0$  and  $T$  [21].

Let us write explicitly Eq. (2.80) substituting the expressions (2.25) and (2.76) for  $n(n_0, T)$  and  $n_n(n_0, T)$ , obtaining

$$n_s = n_0 + \int \frac{d^D k}{(2\pi)^D} \frac{\frac{\hbar^2 k^2}{2m} + n_0 V(k)}{E_k(n_0)} \left[ \frac{1}{2} + \frac{1}{e^{\beta E_k(n_0)} - 1} \right] - \frac{\beta}{mD} \int \frac{d^D k}{(2\pi)^D} \hbar^2 k^2 \frac{e^{\beta E_k(n_0)}}{(e^{\beta E_k(n_0)} - 1)^2} \quad (2.82)$$

For the sake of completeness we remind that the number density corresponds to the first three terms, namely

$$n(n_0, T) = n_0 + \int \frac{d^D k}{(2\pi)^D} \frac{\frac{\hbar^2 k^2}{2m} + n_0 V(k)}{E_k(n_0)} \left[ \frac{1}{2} + \frac{1}{e^{\beta E_k(n_0)} - 1} \right] \quad (2.83)$$

In the next chapter we will implement Eq. (2.82) in  $D = 3$ , explicitly substituting the interaction potential  $V(\vec{k})$  with a zero-range interaction or a finite-range effective interaction.

Depending on the specific case, it will also be necessary to express the superfluid density  $n_s$  as a function of the density  $n$  and the temperature  $T$  as

$$n_s = n - \frac{\beta}{mD} \int \frac{d^D k}{(2\pi)^D} \hbar^2 k^2 \frac{e^{\beta E_k(n_0(n,T))}}{(e^{\beta E_k(n_0(n,T))} - 1)^2} \quad (2.84)$$

in which the number density  $n(n_0, T)$  of Eq. (2.83) is used to express the condensate density  $n_0 = n_0(n, T)$  as a function of the density  $n$ . In particular, we will discuss the two-dimensional case, in which we will show that the thermal contribution to the density  $f_g^{(T)}$  is divergent. In that case, the superfluid density  $n_s$  is obtained as

$$n_s = n - \frac{\beta}{mD} \int \frac{d^D k}{(2\pi)^D} \hbar^2 k^2 \frac{e^{\beta E_k(n_0(n,T=0))}}{(e^{\beta E_k(n_0(n,T=0))} - 1)^2} \quad (2.85)$$

namely expressing  $n_0 = n_0(n, T = 0)$  from the zero-temperature number density of Eq. (2.21). We expect this to represent a reliable equation for  $n_s$  as a function of  $n$  and  $T$  in the low-temperature regime.

To test the consistency of the equations obtained in this section, one can substitute  $D = 3$  and  $n_0 = 0$  in the Eq. (2.82). We expect that, if all particles are out of the condensate, the only non-zero term of the density  $n(n_0, T)$  is the thermal contribution  $f_g^{(T)}$ . Moreover, the system cannot be superfluid, so that  $f_g^{(T)} = n_n$ , hence the superfluid density must be zero  $n_s = 0$ . This is exactly what happens, since the superfluid density becomes

$$n_s(n_0 = 0, T) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{e^{\beta \frac{\hbar^2 k^2}{2m}} - 1} - \frac{\beta}{3m} \int \frac{d^3 k}{(2\pi)^3} \hbar^2 k^2 \frac{e^{\beta \frac{\hbar^2 k^2}{2m}}}{(e^{\beta \frac{\hbar^2 k^2}{2m}} - 1)^2} \quad (2.86)$$

While the first integral can be deleted thank to the Veltman conjecture (1.73), it can be easily shown that extending the third integral to spherical coordinates and performing an integration by parts, the last two integrals are the same quantity  $f_g^{(T)}(n_0) = n_n(n_0, T)$ . As we expected, this implies that a three-dimensional bosonic system, in absence of the condensate, has also  $n_s = 0$ .

## Chapter 3

# Explicit formulation for $D$ -dimensional interacting bosons

In the previous chapter we have obtained general equations for  $D$ -dimensional bosons in terms of a generic interaction potential  $V(\vec{k})$ . In the first section we will treat the zero-range interaction  $V(\vec{k}) = g_0$ , while in the second we will extend the derivation to the finite-range interaction  $V(\vec{k}) = g_0 + g_2 k^2$ . In both cases we will derive explicit formulas for the grand potential, the system density in terms of the condensate density and the superfluid density for three- and two- dimensional systems.

### 3.1 Zero-range interaction in $D$ dimensions

Let us consider bosons interacting with the zero-range interaction

$$V(\vec{r} - \vec{r}') = g_0 \delta(\vec{r} - \vec{r}') \quad (3.1)$$

where  $g_0$  is the interaction strength (1.51). We also refer to this interaction in terms of *contact* interaction, since it describe bosons which interact only if are in the same position.

This zero-range approximation of the real interaction potential is quite reliable in the description of ultracold and dilute gases made of alkali-metal atoms, where the interatomic distance  $d$  is much larger than the characteristic range of the interatomic potential [32]. The Fourier transform of the contact interaction is given by, according to its definition (1.40)

$$V(\vec{k}) = g_0 \quad (3.2)$$

Since the general treatment of section (1.3) is valid for a generic interaction  $V(\vec{k})$ , we explicitly calculate the general excitation spectrum substituting the Fourier transform

of the interaction (3.2) into Eq. (1.65), so that

$$E_k = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu + 2g_0\psi_0^2\right)^2 - (g_0\psi_0^2)^2} \quad (3.3)$$

Notice that, to derive the number density  $n(n_0, T)$  and the normal density  $n_n(n_0, T)$  in the context of the microscopic model of section (2.4), one needs to substitute in the excitation spectrum the chemical potential  $\mu$  with the effective chemical potential  $\mu_{\text{eff}}$  and identify  $\mu_{\text{eff}} = g_0\psi_0^2 = g_0n_0$  only after the calculation of the derivative of the grand potential  $\Omega$ . After these operations, according to Eq. (2.77), one obtains the excitation spectrum  $E_k(n_0)$

$$E_k(n_0) = \sqrt{\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + 2n_0g_0\right)} \quad (3.4)$$

Differently, to derive the thermodynamic properties of the system, it is necessary to substitute in the excitation spectrum the mean field value of the order parameter given by the saddle point condition  $\psi_0^2 = \mu/g_0$  of Eq. (2.2). For bosons with zero-range interaction, this leads to the Bogoliubov excitation spectrum [8]

$$E_{k,\text{Bogoliubov}} = \sqrt{\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + 2\mu\right)} \quad (3.5)$$

A remarkable property of the Bogoliubov spectrum  $E_{k,\text{Bogoliubov}}$  is that it is phononic for low wavenumbers, while it approaches a free-particle spectrum for large wavenumbers. The Bogoliubov spectrum has been plotted in Fig. (3.1).

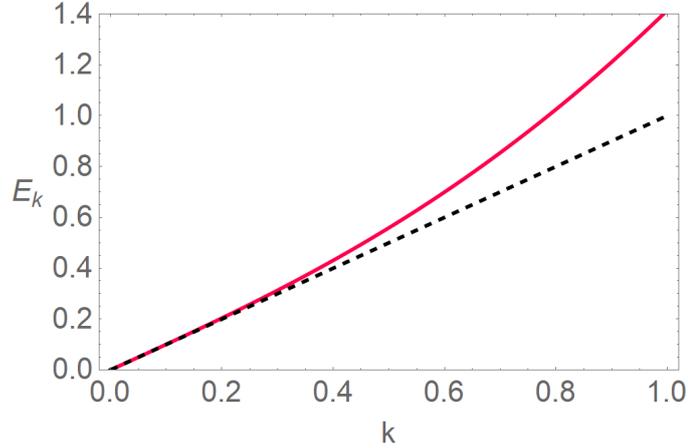


Figure 3.1: Bogoliubov spectrum  $E_{k,\text{Bogoliubov}}$  (red solid line), where the energy is plotted in unity of  $2\mu$ , while the wavevector is rescaled in unity of  $\hbar/(4m\mu)$ . Notice that the Bogoliubov spectrum approaches a phononic dispersion law (black dashed line) for low values of the wavevector  $k$ , while it grows quadratically for large  $k$ , namely as a free-particle spectrum.

According to the Landau criterion (2.33) the fact that the excitation spectrum is linear for  $k \rightarrow 0$  implies that we are describing a superfluid system. In particular, the system shows a superfluid flow for fluid velocities under the sound velocity  $u$ , given by

$$u = \sqrt{\frac{\hbar^2 \mu}{m}} \quad (3.6)$$

### 3.1.1 Thermodynamic properties

We now study the thermodynamic properties of bosons with zero-range interaction, implementing the equations obtained in the section (2.1). Since it is fundamental to derive all the other thermodynamical functions, we will focus on the calculation of the grand potential per unit of volume  $\Omega/L^D$ , given at a gaussian level by Eqs. (2.4) and (2.5), obtaining explicit results in  $D = 3$  and  $D = 2$ .

Regarding the order parameter grand potential  $\Omega_0/L^D$ , since it does not depend on  $V(\vec{k})$ , it is given by Eq. (2.4), namely

$$\frac{\Omega_0}{L^D} = -\frac{\mu^2}{2g_0} \quad (3.7)$$

We focus on the gaussian contribution  $\Omega_g$  of the grand potential (1.66) which consists of two terms: let us first consider the zero-temperature term. Employing the Bogoliubov spectrum  $E_{k,\text{Bogoliubov}}$  of Eq. (3.5), we express the zero-temperature gaussian grand potential per unit of volume  $\Omega_g^{(0)}/L^D$  in the thermodynamic limit as

$$\frac{\Omega_g^{(0)}}{L^D} = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2\mu \right)} \quad (3.8)$$

Since the integrand function depends only on the modulus of the wavevector  $|\vec{k}|$ , we calculate the integral using  $D$ -dimensional spherical coordinates, namely

$$\frac{\Omega_g^{(0)}}{L^D} = \frac{S_D}{2(2\pi)^D} \int_0^{+\infty} dk k^{D-1} \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2\mu \right)} \quad (3.9)$$

Changing the integration variable to  $t = \hbar^2 k^2 / (4m\mu)$  the integral can be recast in the adimensional form

$$\frac{\Omega_g^{(0)}}{L^D} = \frac{S_D}{(2\pi)^D} \left( \frac{m}{\hbar^2} \right)^{D/2} \mu^{\frac{D+2}{2}} \int_0^{+\infty} dt t^{\frac{D-1}{2}} (1+t)^{1/2} \quad (3.10)$$

The last integral is evidently divergent, though it can be regularized following the dimensional regularization procedure described in section (1.4). The first step consists in extending the integral to the complex dimension  $\mathcal{D} = D - \varepsilon$ , so that

$$\frac{\Omega_g^{(0)}}{L^D} = \frac{\kappa^\varepsilon \mu}{\Gamma[\mathcal{D}/2]} \left( \frac{m\mu}{\pi\hbar^2} \right)^{\mathcal{D}/2} \int_0^{+\infty} dt t^{\frac{\mathcal{D}-1}{2}} (1+t)^{1/2} \quad (3.11)$$

where the wavevector  $\kappa$  has been introduced for dimensional reasons, due to the extension to complex dimension  $\mathcal{D}$ . The integral can now be rewritten in terms of the Euler beta function (1.75), such as, using the property of Eq. (1.77) and the  $\varepsilon \rightarrow 0$  expansion (1.81), the zero-temperature grand potential per unit of volume  $\Omega_g^{(0)}/L^D$  reads

$$\frac{\Omega_g^{(0)}}{L^D} = -\frac{\mu^{\frac{D+2}{2}}}{2\pi^{\frac{D+1}{2}}} \left(\frac{m}{\hbar^2}\right)^{D/2} \left[1 + \frac{\varepsilon}{2} \ln\left(\frac{\pi\hbar^2\kappa^2}{m\mu}\right) + o(\varepsilon^2)\right] \frac{\Gamma[(D-\varepsilon+1)/2] \Gamma[(-D+\varepsilon-2)/2]}{\Gamma[(D-\varepsilon)/2]} \quad (3.12)$$

Since the form of Eq. (3.12) changes depending on the spatial dimension  $D$ , before substituting  $D$  we calculate the thermal gaussian contribution to the grand potential.

According to the second term of Eq. (2.5), we write the thermal gaussian grand potential per unit of volume  $\Omega_g^{(T)}/L^D$  in the thermodynamic limit as

$$\frac{\Omega_g^{(T)}}{L^D} = \frac{1}{\beta} \int \frac{d^D k}{(2\pi)^D} \ln\left(1 - e^{-\beta E_{k,\text{Bogoliubov}}}\right) \quad (3.13)$$

As we have already mentioned, the fact that the Bogoliubov spectrum  $E_{k,\text{Bogoliubov}}$  depends only on  $|\vec{k}|$  allows us to use the  $D$ -dimensional spherical coordinates (1.72), such as

$$\frac{\Omega_g^{(T)}}{L^D} = \frac{S_D}{\beta(2\pi)^D} \int_0^{+\infty} dk k^{D-1} \ln\left(1 - e^{-\beta E_{k,\text{Bogoliubov}}}\right) \quad (3.14)$$

An integration by parts leads to the expression

$$\frac{\Omega_g^{(T)}}{L^D} = -\frac{1}{2^{D-1}\pi^{D/2}\Gamma[D/2]} \frac{1}{D} \int_0^{+\infty} dk k^D \frac{\partial E_{k,\text{Bogoliubov}}}{\partial k} \frac{1}{e^{\beta E_{k,\text{Bogoliubov}}} - 1} \quad (3.15)$$

where we have substituted the explicit form of  $S_D$ . Unlike the zero-temperature contribution (3.12),  $\Omega_g^{(T)}/L^D$  is convergent. However, this integral can be calculated analytically only in the low-temperature regime, in which it is useful to introduce the variable  $x = \beta E_{k,\text{Bogoliubov}}$ . The gaussian finite-temperature grand potential correction now becomes

$$\frac{\Omega_g^{(T)}}{L^D} = -\frac{1}{2^{D-1}\pi^{D/2}\Gamma[D/2]} \frac{1}{\beta D} \int_0^{+\infty} dx k(x)^D \frac{1}{e^x - 1} \quad (3.16)$$

where we express  $k(x)$  as a function of  $x$  as

$$k(x) = \sqrt{\frac{2m\mu}{\hbar^2}} \sqrt{-1 + \sqrt{1 + \frac{x^2}{\mu^2\beta^2}}} \quad (3.17)$$

The integral in Eq. (3.16) can be explicitly performed by expanding the function  $k(x)$  in the low-temperature regime. Since the integrand depends on the system dimension  $D$ , we need to explicitly substitute it. In the following we calculate the zero-temperature gaussian grand potential per unit of volume  $\Omega_g^{(0)}/L^D$  and the thermal gaussian grand potential per unit of volume  $\Omega_g^{(T)}/L^D$  first in  $D = 3$ , then in  $D = 2$ .

**D=3**

To calculate the zero-temperature gaussian grand potential in the three-dimensional case, the extension to complex dimension is not really necessary, since the gamma functions involved converge to finite values. For this reason, it is sufficient to put  $D = 3$  and directly substitute  $\varepsilon = 0$  in equation (3.12), obtaining

$$\frac{\Omega_g^{(0)}}{L^3} = \frac{8}{15\pi^2} \left(\frac{m}{\hbar^2}\right)^{3/2} \mu^{5/2} \quad (3.18)$$

The thermal gaussian grand potential can be calculated by expanding the integrand of (3.16) in the low-temperature limit and choosing  $D = 3$ . A simple integration leads to

$$\frac{\Omega_g^{(T)}}{L^3} = -\frac{\pi^2}{90} \left(\frac{m}{\hbar^2}\right)^{3/2} \frac{(k_B T)^4}{\mu^{3/2}} + o((k_B T)^6) \quad (3.19)$$

The beyond mean field grand potential per unit of volume  $\Omega/L^3$ , up to gaussian corrections then becomes

$$\frac{\Omega}{L^3} = -\frac{\mu^2}{2g_0} + \frac{8}{15\pi^2} \left(\frac{m}{\hbar^2}\right)^{3/2} \mu^{5/2} - \frac{\pi^2}{90} \left(\frac{m}{\hbar^2}\right)^{3/2} \frac{(k_B T)^4}{\mu^{3/2}} + o((k_B T)^6) \quad (3.20)$$

which is in full agreement with previously known results [10].

**D=2**

In the two-dimensional case dimensional regularization itself is not sufficient and  $\Omega_g^{(0)}/L^2$  is still divergent due to the presence of  $\Gamma[-2]$  in Eq. (3.12) where  $D = 2$  is substituted. This divergence can be removed performing all the steps described in section (1.4), i. e. expanding the various functions for  $\varepsilon \rightarrow 0$ . Doing so one obtains

$$\frac{\Omega_g^{(0)}}{L^2} = -\frac{m\mu^2}{2\pi^{3/2}\hbar^2} \left[ \frac{\pi^{1/2}}{2} \frac{1}{\varepsilon} + \frac{\pi^{1/2}}{8} (\ln(16) - 2\gamma - 1) + \frac{\pi^{1/2}}{4} \ln\left(\frac{\pi\hbar^2\kappa^2}{m\mu}\right) + o(\varepsilon) \right] \quad (3.21)$$

In the renormalization spirit of "extracting finite information from infinities" [49] we delete the diverging  $o(\varepsilon^{-1})$  term and rearrange the remaining terms in the form

$$\frac{\Omega_g^{(0)}}{L^2} = -\frac{m\mu^2}{8\pi\hbar^2} \left[ \ln\left(\frac{4\pi\hbar^2\kappa^2}{m\mu e^\gamma}\right) - \frac{1}{2} \right] \quad (3.22)$$

We also add the order parameter grand potential, obtaining the zero-temperature part of the grand potential

$$\frac{\Omega_0}{L^2} + \frac{\Omega_g^{(0)}}{L^2} = -\frac{\mu^2}{2g_0} - \frac{m\mu^2}{8\pi\hbar^2} \left[ \ln\left(\frac{4\pi\hbar^2\kappa^2}{m\mu e^\gamma}\right) - \frac{1}{2} \right] \quad (3.23)$$

The scale  $\kappa$  can be fixed comparing this equation with the zero-temperature pressure derived by Popov [41] and identifying the energy cutoff

$$\epsilon_0 = \frac{4\pi\hbar^2\kappa^2}{m \exp\left(\gamma - \frac{4\pi\hbar^2}{mg_0}\right)} \quad (3.24)$$

In this way we obtain the zero-temperature part of the grand potential per unit of area in the renormalized form

$$\frac{\Omega_0}{L^2} + \frac{\Omega_g^{(0)}}{L^2} = -\frac{m\mu^2}{8\pi\hbar^2} \left[ \ln\left(\frac{\epsilon_0}{\mu}\right) - \frac{1}{2} \right] \quad (3.25)$$

Notice that in reference [37] the authors obtained a two-loop (beyond gaussian) correction that goes as  $[\ln(\epsilon_0/\mu)]^{-1}$ .

Like in the three-dimensional case, we can calculate also the thermal gaussian contribution  $\Omega_g^{(T)}/L^2$  expanding the integrand of the equation (3.16) in the zero-temperature limit and choosing  $D = 2$ , obtaining

$$\frac{\Omega_g^{(T)}}{L^2} = -\frac{\zeta(3)}{2\pi} \left(\frac{m}{\hbar^2}\right) \frac{(k_B T)^3}{\mu} + o((k_B T)^5) \quad (3.26)$$

Finally, we get the renormalized two-dimensional grand potential calculated up to gaussian corrections

$$\frac{\Omega}{L^2} = -\frac{m\mu^2}{8\pi\hbar^2} \left[ \ln\left(\frac{\epsilon_0}{\mu}\right) - \frac{1}{2} \right] - \frac{\zeta(3)}{2\pi} \left(\frac{m}{\hbar^2}\right) \frac{(k_B T)^3}{\mu} + o((k_B T)^5) \quad (3.27)$$

### 3.1.2 Quantum depletion and thermal condensate fraction

We now explicitly implement the Eq. (2.21) for the zero-temperature density  $n(n_0, T = 0)$  in the case of bosons with zero-range interaction. Since the system density is given by the expression

$$n = n_0 + f_g^{(0)}(n_0) \quad (3.28)$$

we need to calculate the zero-temperature density contribution  $f_g^{(0)}(n_0)$ , given by Eq. (2.24) in which the zero-range interaction  $V(\vec{k}) = g_0$  is substituted, namely

$$f_g^{(0)}(n_0) = \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \frac{\frac{\hbar^2 k^2}{2m} + g_0 n_0}{\sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2g_0 n_0 \right)}} \quad (3.29)$$

Using  $D$ -dimensional spherical coordinates we rewrite the previous quantity as

$$f_g^{(0)}(n_0) = \frac{S_D}{2(2\pi)^D} \int_0^{+\infty} dk k^{D-1} \frac{\frac{\hbar^2 k^2}{2m} + g_0 n_0}{\sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2g_0 n_0 \right)}} \quad (3.30)$$

Notice that this integral is ultraviolet divergent, but can be regularized following the standard procedure of dimensional regularization outlined in section (1.4). First of all, as we have done for the grand potential in the previous section, we introduce the integration variable  $t = \hbar^2 k^2 / (4mg_0 n_0)$  to get an adimensional integral and we extend the system spatial dimension  $D$  to the complex dimension  $\mathcal{D} = D - \varepsilon$ . We obtain

$$f_g^{(0)}(n_0) = \frac{\kappa^\varepsilon}{2\Gamma(\mathcal{D}/2)} \left( \frac{mg_0 n_0}{\pi \hbar^2} \right)^{\mathcal{D}/2} \int_0^{+\infty} dt \frac{t^{\frac{\mathcal{D}-1}{2}} + t^{\frac{\mathcal{D}-3}{2}}}{(1+t)^{1/2}} \quad (3.31)$$

The divergent integral is then regularized remembering the property (1.77) of the beta function, using the recursive relation (1.76) for the gamma function and expanding the result for low values of  $\varepsilon$ , such as the zero-temperature density contribution  $f_g^{(0)}(n_0)$  reads

$$f_g^{(0)}(n_0) = \frac{(D-2-\varepsilon)}{8\pi^{\frac{D+1}{2}}} \left( \frac{mg_0 n_0}{\hbar^2} \right)^{D/2} \left[ 1 + \frac{\varepsilon}{2} \ln \left( \frac{\pi \hbar^2 \kappa^2}{mg_0 n_0} \right) + o(\varepsilon^2) \right] \frac{\Gamma[(\varepsilon-D)/2] \Gamma[(D-\varepsilon-1)/2]}{\Gamma[(D-\varepsilon)/2]} \quad (3.32)$$

Before proceeding to explicit calculation of this contribution in  $D = 3$  and  $D = 2$ , we also calculate the thermal contribution  $f_g^{(T)}(n_0)$  to the system density  $n(n_0, T)$  of Eq. (2.25). Differently from the zero-temperature contribution,  $f_g^{(T)}(n_0)$  is convergent and for a system of bosons with zero-range interaction is given by

$$f_g^{(T)}(n_0) = \frac{S_D}{(2\pi)^D} \int_0^{+\infty} dk \frac{k^{D-1}}{e^{\beta E_k(n_0)} - 1} \frac{\frac{\hbar^2 k^2}{2m} + n_0 g_0}{E_k(n_0)} \quad (3.33)$$

where  $E_k(n_0)$  is the excitation spectrum (3.4). To perform the analytical calculation of the integral we use the variable  $x = \beta E_k(n_0)$ , in terms of which the gaussian finite-temperature density correction becomes

$$f_g^{(T)}(n_0) = \frac{S_D}{(2\pi)^D k_B T} \int_0^{+\infty} dx \frac{dk(x)}{dx} \frac{k(x)^{D-1}}{x(e^x - 1)} \left( \frac{\hbar^2 k(x)^2}{2m} + n_0 g_0 \right) \quad (3.34)$$

where

$$k(x) = \sqrt{\frac{2mn_0 g_0}{\hbar^2}} \sqrt{-1 + \sqrt{1 + \frac{(k_B T)^2 x^2}{n_0^2 g_0^2}}} \quad (3.35)$$

The calculation of  $f_g^{(T)}(n_0)$  can be performed analytically only in the low temperature limit, in which we will expand the integrand after explicitly choosing the spatial dimension  $D$ .

Let us now substitute  $D = 3$  and  $D = 2$  in the equations (3.32) and (3.34).

### D=3

To obtain the three-dimensional value of the zero-temperature density contribution  $f_g^{(0)}(n_0)$ , it is sufficient to substitute  $D = 3$  and  $\varepsilon = 0$  into the equation (3.32), obtaining

$$f_g^{(0)}(n_0) = \frac{1}{3\pi^2} \left( \frac{mg_0 n_0}{\hbar^2} \right)^{3/2} \quad (3.36)$$

Hence, the zero-temperature equation for the number density (2.21) is

$$n = n_0 + \frac{1}{3\pi^2} \left( \frac{mg_0 n_0}{\hbar^2} \right)^{3/2} \quad (3.37)$$

With the scattering theory [47] the contact interaction strength  $g_0$  is related to the s-wave scattering length  $a_s$  by

$$g_0 = \frac{4\pi\hbar^2 a_s}{m} \quad (3.38)$$

It follows that the zero-temperature total density  $n$  is related to the condensate density  $n_0$  through the formula

$$n = n_0 \left[ 1 + \frac{8}{3\sqrt{\pi}} (n_0 a_s^3)^{1/2} \right] \quad (3.39)$$

Our result is consistent at the order  $o((na_s^3)^{1/2})$  with the Bogoliubov formula for quantum depletion [8]. In fact, approximating the condensate density  $n_0$  with  $n$  in the second term in the square bracket and expanding  $n_0/n$  around small values of the gas parameter  $(na_s^3)^{1/2}$  we reproduce Bogoliubov's result

$$\frac{n_0}{n} = 1 - \frac{8}{3\sqrt{\pi}} (na_s^3)^{1/2} \quad (3.40)$$

Since we are studying weakly-interacting bosons, for which it is assumed that the three-dimensional gas parameter is  $na_s^3 \ll 1$ , we expect that condensate fraction  $n_0/n$  obtained from our Eq. (3.39) is consistent.

The plot of the condensate fraction  $n_0/n$  at zero temperature can be made by rewriting Eq. (3.39) in terms of the rescaled density  $\tilde{n}$  and the rescaled condensate density  $\tilde{n}_0$  as

$$\tilde{n} = na_s^3 \qquad \tilde{n}_0 = n_0 a_s^3 \qquad (3.41)$$

in terms of which the relation (3.39) becomes

$$\tilde{n} = \tilde{n}_0 + \frac{8}{3\sqrt{\pi}} \tilde{n}_0^{3/2} \qquad (3.42)$$

Solving numerically the previous equation and obtaining the density  $\tilde{n}$  value for many different values of  $\tilde{n}_0$  one can evaluate the condensate fraction  $n_0/n = \tilde{n}_0/\tilde{n}$  for many values of the gas parameter  $na_s^3 = \tilde{n}$ .

In Fig. 3.2 we compare Eq. (3.39), blue solid line, with Eq. (3.40), red dashed line, plotting the condensate fraction  $n_0/n$  in terms of the gas parameter  $na_s^3$ . Notice that, in the weakly-interacting regime in which the gas parameter is  $na_s^3 \approx 10^{-5}$ , the condensate fraction is around 0.99. This result is in good agreement with the experimental results for cold alkali-atoms systems, for which the contact interaction approximation is good, while, in the case of He-II in which the condensate fraction is less than 10%, this framework is not reliable.

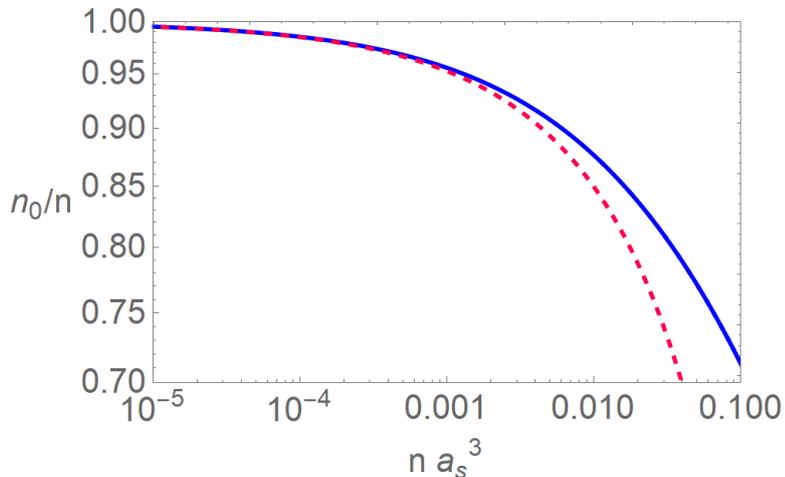


Figure 3.2: Zero-temperature condensate fraction  $n_0/n$  for three-dimensional bosons with contact interaction, plotted as a function of the gas parameter  $na_s^3$ . Blue solid line: our condensate fraction, obtained from Eq. (3.39). Red dashed line: Bogoliubov condensate fraction (3.40). Notice that the curves are coincident for low values of the gas parameter, in the physical regime where it is expected that Eqs. (3.39) and (3.40) are equivalent.

Now we consider also the finite-temperature density contribution  $f_g^{(T)}(n_0)$ , which can be obtained expanding the integrand of equation (3.34) for low temperatures and performing

the integral in this limit, in which it is convergent. We find

$$f_g^{(T)}(n_0) = \frac{(k_B T)^2}{12(n_0 g_0)^{1/2}} \left( \frac{m}{\hbar^2} \right)^{3/2} \left( 1 - \frac{\pi^2 (k_B T)^2}{20(n_0 g_0)^2} \right) + o((k_B T)^5) \quad (3.43)$$

Putting the contributions of Eqs. (3.42) and (3.43) inside Eq. (2.25), we write the number density  $n$  for contact-interacting bosons in the low-temperature limit as

$$n = n_0 + \frac{1}{3\pi^2} \left( \frac{m g_0 n_0}{\hbar^2} \right)^{3/2} + \frac{(k_B T)^2}{12(n_0 g_0)^{1/2}} \left( \frac{m}{\hbar^2} \right)^{3/2} \quad (3.44)$$

We now use this expression to calculate the thermal condensate fraction  $n_0/n$ . First of all we substitute the three-dimensional contact interaction strength  $g_0$  (expressed in terms of the scattering length  $a_s$  with scattering theory in Eq. (3.38)), rewriting the previous density equation as a function of the gas parameter  $na_s^3$  in the equivalent form

$$1 = \left( \frac{n_0}{n} \right) + \frac{8}{3\sqrt{\pi}} (na_s^3)^{1/2} \left( \frac{n_0}{n} \right)^{3/2} + \frac{1}{24\sqrt{\pi}} \frac{(k_B T)^2}{(na_s^3)^{1/6}} \left( \frac{m}{\hbar^2 n^{2/3}} \right)^2 \left( \frac{n}{n_0} \right)^{1/2} \quad (3.45)$$

We approximate  $(n_0/n)^{3/2} \approx 1$  in the second term and  $(n_0/n)^{1/2} \approx 1$  in the third term at the right-hand side, since, under the hypothesis that the bosons are very weakly-interacting and for low temperatures, these terms constitute a small correction of the first. Then we obtain the thermal condensate fraction for very low values of the gas parameter and temperatures, namely

$$\frac{n_0}{n} = 1 - \frac{8}{3\sqrt{\pi}} (na_s^3)^{1/2} \left[ 1 + \frac{1}{64} \frac{(k_B T)^2}{(na_s^3)^{2/3}} \left( \frac{m}{\hbar^2 n^{2/3}} \right)^2 \right] \quad (3.46)$$

However, notice that a general result for the thermal condensate fraction  $n_0/n$  can only be obtained numerically: in order to plot it we introduce the adimensional rescaled temperature

$$T^* = \frac{k_B T}{E_r} \quad (3.47)$$

where  $E_r$  is an arbitrary reference energy given by

$$E_r = \frac{\hbar^2}{m} n^{2/3} \quad (3.48)$$

such as the Eq. (3.45) can be expressed in the implicit form

$$\left( \frac{n_0}{n} \right) + \frac{8}{3\sqrt{\pi}} (na_s^3)^{1/2} \left( \frac{n_0}{n} \right)^{3/2} + \frac{1}{24\sqrt{\pi}} \frac{T^{*2}}{(na_s^3)^{1/6}} \left( \frac{n}{n_0} \right)^{1/2} - 1 = 0 \quad (3.49)$$

In Fig. (3.3) we represent the numeric solution of Eq. (3.49) for different values of the gas parameter  $na_s^3$ .

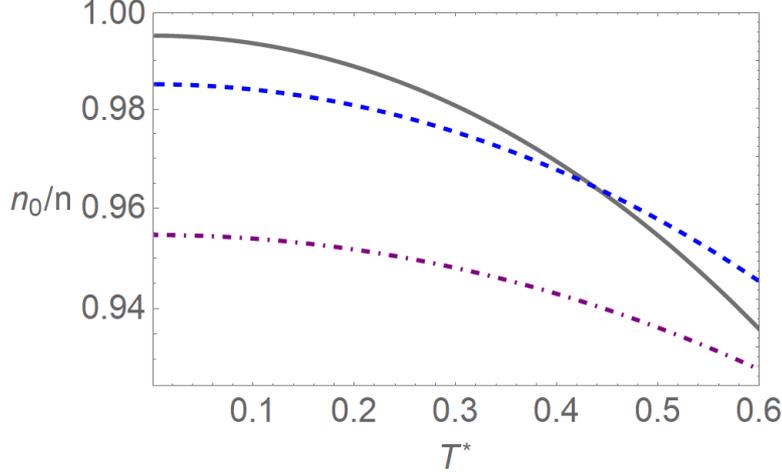


Figure 3.3: Condensate fraction  $n_0/n$  for bosons in  $D = 3$  with contact interaction. The condensate fraction is represented as a function of the adimensional temperature  $T^* = k_B T/E_r$ , rescaled with the arbitrary energy scale  $E_r = \hbar^2 n^{2/3}/m$ . The three curves are obtained as the numerical solution of Eq. (3.49) with the explicit choice of the three-dimensional gas parameter  $na_s^3$  values:  $10^{-5}$  (grey solid line),  $10^{-4}$  (blue dashed line),  $10^{-3}$  (purple dot-dashed line)

## D=2

In the two-dimensional case, the zero-temperature density contribution  $f_g^{(0)}$  of Eq. (3.32) can be calculated taking into account the limit

$$\lim_{D \rightarrow 2, \varepsilon \rightarrow 0} (D - 2 - \varepsilon) \Gamma[(\varepsilon - D)/2] = 2 \quad (3.50)$$

Therefore  $f_g^{(0)}$  does not show the  $o(\varepsilon^{-1})$  divergence associated to the gamma function  $\Gamma[(\varepsilon - D)/2]$  in  $D = 2$ , and the number density (2.21) reads

$$n = n_0 + \frac{1}{4\pi} \frac{m}{\hbar^2} g_0 n_0 \quad (3.51)$$

In two spatial dimensions the relationship between the strength  $g_0$  and the two-dimensional s-wave scattering length  $a_s$  is more involved than in the three-dimensional case. At the lowest order in the two-dimensional gas parameter  $na_s^2$  the two-dimensional zero-range interaction strength  $g_0$  is given by [6]

$$g_0 = \frac{4\pi\hbar^2}{m} \frac{1}{|\ln(na_s^2)|} \quad (3.52)$$

Consequently the zero-temperature total number density  $n(n_0, T = 0)$  can be written in terms of the condensate density  $n_0$  as

$$n = n_0 + \frac{n_0}{|\ln(na_s^2)|} \quad (3.53)$$

Notice that this equation is equivalent, in the weakly-interacting regime in which the gas parameter  $na_s^2 \ll 1$ , to the familiar calculation of the condensate density made by Schick, which reads [46]

$$\frac{n_0}{n} = 1 - \frac{1}{|\ln(na_s^2)|} \quad (3.54)$$

Analogously to the three-dimensional case, one can define the rescaled number density and quasi-condensate rescaled number density

$$\tilde{n} = na_s^2 \quad \tilde{n}_0 = n_0 a_s^2 \quad (3.55)$$

in terms of which we calculate the condensate fraction  $\tilde{n}_0/\tilde{n}$  rewriting Eq. (3.53) as

$$\frac{\tilde{n}_0}{\tilde{n}} = \frac{1}{1 + \frac{1}{|\ln(\tilde{n})|}} \quad (3.56)$$

In Fig. (3.4) we plot the condensate fraction  $n_0/n = \tilde{n}_0/\tilde{n}$  as a function of the two-dimensional gas parameter given by the numerical solution of Eq. (3.56) and the numerical condensate fraction obtained from the Schick result of Eq. (3.54). The comparison between this plot and the analogous in  $D = 3$  of Fig. (3.2) shows that the depletion of the condensate, taken at the same values of the gas parameter, is higher in the two-dimensional case.

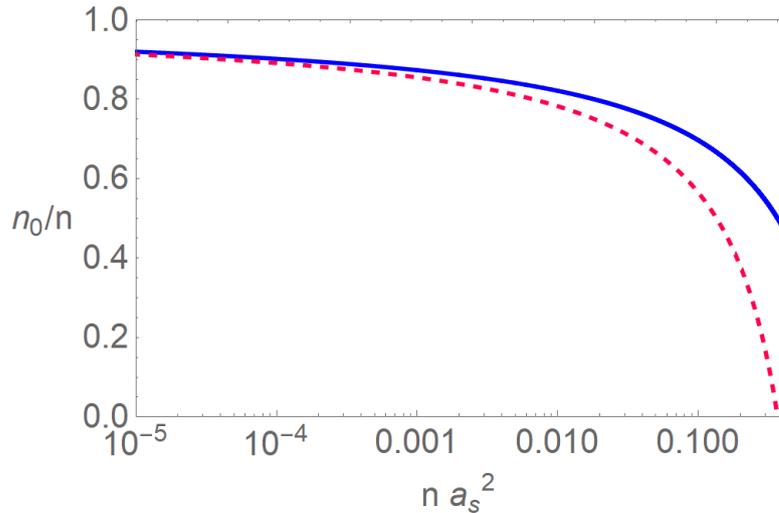


Figure 3.4: Zero-temperature condensate fraction  $n_0/n$  as a function of the gas parameter  $na_s^2$  for bosonic cold atoms in  $D = 2$  interacting with the contact interaction (3.2). In the blue solid line we represent the numerical solution of Eq. (3.56), while in the red dashed line we plot the Schick relation (3.54). Notice that the curves are coincident for low value of the two-dimensional gas parameter  $na_s^2$ , which constitutes the physical regime in which we expect that our approach holds.

As in the three-dimensional case, we would like to calculate the finite-temperature density contribution  $f_g^{(T)}(n_0)$ . However, substituting  $D = 2$  in Eq. (3.34), we find the

integral

$$f_g^{(T)}(n_0) = \frac{mk_B T}{2\pi\hbar^2} \int_0^{+\infty} dx \frac{1}{e^x - 1} \quad (3.57)$$

which is infrared divergent, namely it diverges in the low energy part of the spectrum. This kind of divergence has a physical meaning and is related to the Mermin-Wagner theorem. The theorem essentially states that in presence of a continuous symmetry, there cannot be long-range-order in one- and two- dimensional systems. Since long-range-order is related to the  $U(1)$  spontaneous symmetry breaking, the symmetry cannot be broken and the ground state of the system shares the same symmetry of the hamiltonian. If spontaneous symmetry breaking would happen, the sum of the long-wavelength fluctuations of the system would diverge, which is exactly what happens with the divergence of  $f_g^{(T)}(n_0)$ . In other words, Bose-Einstein condensation cannot occur in low-dimensional systems at finite temperature, which is why our density expression in terms of  $n_0$  diverges. In this case one usually refers to quasi-condensation: in a finite-size system (without taking the thermodynamic limit) there will be a finite local condensate fraction, which becomes zero for an infinite system.

### 3.1.3 Superfluid density

Let us remind that the superfluid density  $n_s$ , which we calculate here for bosons with contact interaction, is given by Eq. (2.82) as

$$n_s = n(n_0, T) - n_n(n_0, T) \quad (3.58)$$

Considering that the number density  $n(n_0, T)$  has been calculated in the previous section, we calculate the normal density  $n_n(n_0, T)$  according to Eq. (2.76), namely

$$n_n(n_0, T) = \frac{\beta \hbar^2 S_D}{mD(2\pi)^D} \int_0^{+\infty} dk k^{D+1} \frac{e^{\beta E_k(n_0)}}{(e^{\beta E_k(n_0)} - 1)^2} \quad (3.59)$$

where  $E_k(n_0)$  is the excitation spectrum (3.4). Since the integrand depends only on the modulus of the wavevector  $|\vec{k}|$  we introduce the  $D$ -dimensional spherical coordinates

$$n_n(n_0, T) = \frac{\beta \hbar^2 S_D}{mD(2\pi)^D} \int_0^{+\infty} dk k^{D+1} \frac{e^{\beta E_k(n_0)}}{(e^{\beta E_k(n_0)} - 1)^2} \quad (3.60)$$

In full analogy with the calculation of  $f_g^{(T)}(n_0)$ , we set  $x = \beta E_k(n_0)$ , in terms of which we can express the normal density

$$n_n(n_0, T) = \frac{\beta \hbar^2}{2^{D-1} \pi^{D/2} mD \Gamma[D/2]} \int_0^{+\infty} dx \frac{dk}{dx}(x) k(x)^{D+1} \frac{e^x}{(e^x - 1)^2} \quad (3.61)$$

where  $k(x)$  is given by Eq. (3.35).

As before, we must take the low-temperature limit to perform the integration of Eq. (3.61) and we explicitly consider the  $D = 3$  case and  $D = 2$  case.

#### D=3

In the three-dimensional case we obtain the low-temperature normal density  $n_n(n_0, T)$  as

$$n_n(n_0, T) = \frac{2\pi^2}{45} \left(\frac{m}{\hbar^2}\right)^{3/2} \frac{(k_B T)^4}{(n_0 g_0)^{5/2}} \left(1 - \frac{5\pi^2 (k_B T)^2}{2(n_0 g_0)^2}\right) + o(k_B T)^7 \quad (3.62)$$

It is worth noting that neglecting the  $o((k_B T)^6)$  correction, we reproduce the phononic contribution to normal mass density  $\rho_n$ , as obtained by Landau in the form [29]

$$\rho_n = m n_n = \frac{2\pi^2}{45} \frac{(k_B T)^4}{u^5} \quad (3.63)$$

where, accordingly to Eq. (3.6), the speed of sound  $u$  is now defined as

$$u = \sqrt{\frac{\hbar^2 n_0 g_0}{m}} \quad (3.64)$$

Implementing the superfluid density formula of Eq. (2.82) we obtain, in the low-temperature limit

$$n_s = n_0 + \frac{1}{3\pi^2} \left( \frac{mg_0 n_0}{\hbar^2} \right)^{3/2} + \frac{(k_B T)^2}{12(n_0 g_0)^{1/2}} \left( \frac{m}{\hbar^2} \right)^{3/2} - \frac{7\pi^2}{144} \left( \frac{m}{\hbar^2} \right)^{3/2} \frac{(k_B T)^4}{(n_0 g_0)^{5/2}} \quad (3.65)$$

We employ this equation in the weakly-interacting regime and at very low temperatures in which  $n \approx n_0$  to express the superfluid fraction  $n_s/n$  in terms of the temperature  $T$  and the gas parameter  $na_s^3$  as

$$\frac{n_s}{n} = 1 - \frac{1}{720\sqrt{\pi}} \left( \frac{m}{\hbar^2 n^{2/3}} \right)^4 \frac{(k_B T)^4}{(na_s^3)^{5/6}} \quad (3.66)$$

However, the exact result for the superfluid fraction depends on the condensate fraction  $n_0/n$  and can be expressed as a function of the adimensional temperature  $T^*$  of Eq. (3.47) in the implicit form

$$\frac{n_s}{n} = 1 - \frac{1}{720\sqrt{\pi}} \frac{T^{*4}}{(na_s^3)^{5/6}} \left( \frac{n}{n_0} \right)^{5/2} \quad (3.67)$$

Taking the condensate fraction  $n_0/n$  from the numerical solution of Eq. (3.49) we solve this equation to plot in Fig. (3.5) the superfluid fraction  $n_s/n$  in terms of  $T^*$  for the gas parameter value  $na_s^3 = 10^{-4}$ .

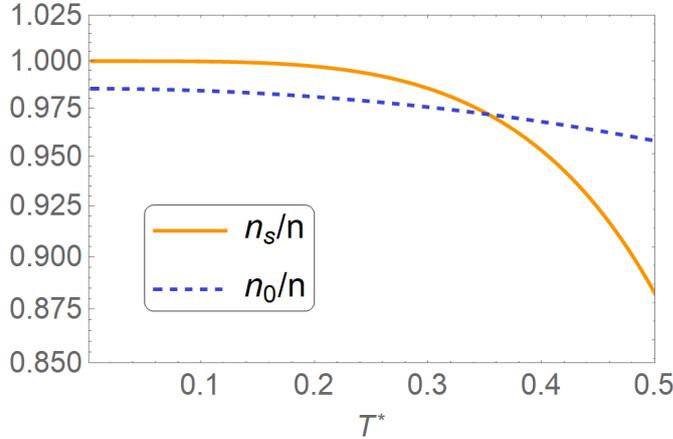


Figure 3.5: Comparison between the superfluid fraction  $n_s/n$  obtained from Eq. (3.67) (orange solid line) and the condensate fraction  $n_0/n$  given by (3.49) (blue dashed line). The curves are represented in  $D = 3$  as functions of the adimensional temperature  $T^* = k_B T/E_r$ , rescaled with the arbitrary energy scale  $E_r = \hbar^2 n^{2/3}/m$ . To make the plot, we choose the gas parameter value  $na_s^3 = 10^{-4}$ . Notice that, at zero temperature  $T = 0$ , the superfluid fraction is equal to 1, while the condensate fraction is lower than 1 because of the quantum depletion of the condensate of interacting bosons.

**D=2**

In the low-temperature limit, the two-dimensional number density is given by Eq. (3.61) as

$$n_n(n_0, T) = \frac{3m}{2\pi\hbar^2} \frac{(k_B T)^3}{(n_0 g_0)^2} \left( \zeta(3) - \frac{15\zeta(5)}{(n_0 g_0)^2} (k_B T)^2 \right) + o(k_B T)^6 \quad (3.68)$$

Since the two-dimensional density contribution  $f_g^{(T)}$  is divergent, we calculate  $n_s$  following the prescriptions of Eq. (2.85), thus expressing the condensate density  $n_0$  as a function of the zero-temperature density  $n(n_0, T=0)$  of Eq. (3.53) and substituting it in  $n_n(n_0, T)$ . Therefore, we find the superfluid density  $n_s$  in the low-temperature regime as a function of  $n$

$$n_s = n - \frac{3\zeta(3)m}{2\pi\hbar^2} \frac{(k_B T)^3}{(n g_0)^2} \left( 1 + \frac{m g_0}{4\pi\hbar^2} \right)^2 \quad (3.69)$$

Notice that, taking the lowest order in the expansion for  $g_0 = 4\pi\hbar^2/(m|\ln(na_s^2)|) \rightarrow 0$  of the previous equation, we get the superfluid fraction  $n_s/n$  as

$$\frac{n_s}{n} = 1 - \frac{3\zeta(3)}{32\pi^3} \left( \frac{m}{\hbar^2 n} \right)^3 |\ln(na_s^2)|^2 (k_B T)^3 \quad (3.70)$$

which is also the result obtained from the Landau formula in the case  $D = 2$ , where  $\mu = g_0 n$ , with  $g_0$  given by Eq. (3.52).

Finally, the exact superfluid fraction can be obtained numerically expressing  $n_s$  of Eq. (3.69) in terms of the two-dimensional rescaled temperature

$$T^* = \frac{k_B T}{E_r} \quad (3.71)$$

where  $E_r = \frac{\hbar^2}{m} n$  is an arbitrary energy scale (notice that  $n$  is the *two*-dimensional density). The superfluid density  $n_s/n$  becomes

$$\frac{n_s}{n} = 1 - \frac{3\zeta(3)}{32\pi^3} (1 + |\ln(na_s^2)|)^2 T^{*3} \quad (3.72)$$

In Fig. (3.6) we plot  $n_s/n$  as a function of the adimensional temperature  $T^*$ , for different choices of the two-dimensional gas parameter value  $na_s^2$ .

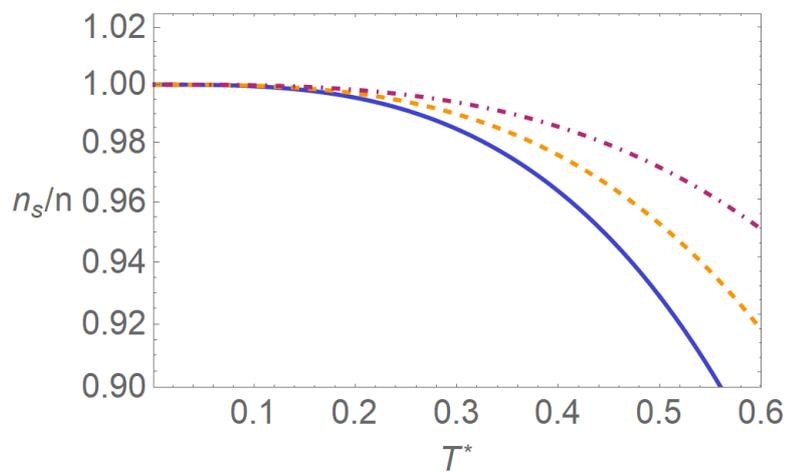


Figure 3.6: Superfluid fraction  $n_s/n$  of a two-dimensional  $D = 2$  system of bosons interacting with the zero-range interaction (3.2). The plot is obtained calculating  $n_s/n$  from Eq. (3.72) as a function of the rescaled temperature  $T^* = k_B T/E_r$ , where  $E_r = \hbar^2 n/m$ . The plot is made for an explicit choice of the gas parameter  $na_s^2$  values:  $10^{-5}$  (blue solid line),  $10^{-4}$  (orange dashed line),  $10^{-3}$  (purple dot-dashed line).



### 3.2 Finite-range effective interaction in $D$ dimensions

The contact zero-range interaction (3.2) studied in the previous section constitutes an approximation in the low-energy limit of the real interaction  $V(\vec{r}, \vec{r}')$  between bosons. Here we try to improve this approximation including a finite-range effective interaction term, which constitutes a small correction of the contact interaction strength  $g_0$ .

The key assumption of this section is that the real space interaction potential depends only on the distance between particles

$$V(\vec{r}, \vec{r}') = V(|\vec{r} - \vec{r}'|) \quad (3.73)$$

Notice that this hypothesis is not general and is suitable only for cold atoms systems in which the anisotropic dipolar effects are negligible. For instance, it cannot be used for the theoretical description of atomic species with a large magnetic momentum like  $^{52}\text{Cr}$ ,  $^{164}\text{Dy}$ ,  $^{168}\text{Er}$ , whose condensation has been achieved in the last years [1, 19, 34]. In fact, the study of such systems requires the inclusion of long-range dipolar effects [7], taken into account by the dipolar interaction

$$V_{dd}(\vec{r} - \vec{r}') = \frac{\mu\mu_0}{4\pi} \frac{1 - 3(\cos\theta)^2}{|\vec{r} - \vec{r}'|^3} \quad (3.74)$$

which is the interaction between polarized bosons with magnetic momentum  $\mu$ , with the angular dependence of the interaction on  $\theta$ , the angle between the polarization direction and  $\vec{r} - \vec{r}'$ .

In the following, we will concentrate on cold atoms systems where the isotropy assumption (3.73) holds. In particular, we want to study the effects of a finite-range correction to the contact interaction of Eq. (3.2). Following a perturbative approach, we take the lowest order correction to the contact interaction in a gradient expansion of the real momentum space interaction potential

$$V(\vec{k}) = g_0 + g_2 k^2 \quad (3.75)$$

We now calculate the form of the coupling constants  $g_0$  and  $g_2$  and we verify that the coupling of the linear term in  $k$  is zero due to Eq. (3.73). Since the atoms interact at a very low energy, we expect that the momentum  $k$  exchanged is almost equal to zero. Therefore, we explicitly calculate the Fourier transform (1.40) of the real space interaction  $V(|\vec{r} - \vec{r}'|) \equiv V(|\vec{r}|)$  in the low momentum limit  $\vec{k} \rightarrow 0$

$$\begin{aligned} V(\vec{k}) &= \int d^D r e^{i\vec{k}\cdot\vec{r}} V(|\vec{r}|) = \\ &= \int d^D r V(|\vec{r}|) + i\vec{k} \cdot \int d^D r \vec{r} V(|\vec{r}|) + \frac{k^2}{2} \int d^D r r^2 V(|\vec{r}|) + \dots \end{aligned} \quad (3.76)$$

As mentioned before, the linear term in  $\vec{k}$  is zero due to the symmetry of the integral, while we define the coupling constants  $g_0$  and  $g_2$  as

$$g_0 = \int d^D r V(|\vec{r}|) \quad (3.77)$$

$$g_2 = \frac{1}{2} \int d^D r r^2 V(|\vec{r}|) \quad (3.78)$$

It has also been shown that the potential (3.75) can be obtained as the Fourier transform of the real space pseudopotential [18]

$$V(\vec{r}, \vec{r}') = g_0 \delta(\vec{r} - \vec{r}') - \frac{g_2}{2} [\vec{\nabla}^2 \delta(\vec{r} - \vec{r}') + \delta(\vec{r} - \vec{r}') \vec{\nabla}^2] \quad (3.79)$$

In this section, we follow the procedures and the results previously outlined, to derive the thermodynamic properties, the number density and the superfluid density of a bosonic system of cold atoms considering the finite-range effective interaction.

Since the general treatment of section (1.3) is valid for a generic interaction  $V(k)$ , it can be particularized in the case of the finite-range interaction (3.75). In this case, the general excitation spectrum (3.3) is given by

$$E_k = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu + \psi_0^2(2g_0 + g_2 k^2)\right)^2 - (\psi_0^2(g_0 + g_2 k^2))^2} \quad (3.80)$$

To study the thermodynamic properties of the system one needs to substitute the mean field value of the uniform field  $\psi_0^2 = \mu/g_0$ , obtaining, for a bosonic system of atoms interacting with the finite-range interaction (3.75), the mean field excitation spectrum

$$E_{k,\text{finite-range}} = \sqrt{\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} \lambda(g_2, \mu/g_0) + 2\mu\right)} \quad (3.81)$$

where

$$\lambda(g_2, \mu/g_0) = 1 + \frac{4m}{\hbar^2} \frac{\mu}{g_0} g_2 \quad (3.82)$$

takes into account the finite-range effects of the interaction. Notice that  $E_{k,\text{finite-range}}$  reduces to the familiar Bogoliubov spectrum (3.5) if the contact interaction is restored by putting  $g_2 = 0$ .

Differently, to obtain the number density  $n(n_0, T)$  and the superfluid density  $n_s$ , one must substitute the chemical potential  $\mu$  with the effective value expressed in terms of the condensate density  $\mu_{\text{eff}} = g_0 \psi_0^2 = g_0 n_0$ , obtaining

$$E_k(n_0) = \sqrt{\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} \lambda(g_2, n_0) + 2g_0 n_0\right)} \quad (3.83)$$

In the next subsections, we follow the same path outlined for the implementation of the equations for zero-range interaction to extend the previous results for a system of bosons with finite-range interaction. Due to this formal analogy, we will not stress all the technical calculation details at the same level of description used before.

### 3.2.1 Thermodynamic properties

Analogously to the contact interaction case, we calculate the grand potential per unit of volume  $\Omega/L^D$ , which can be used to derive the other thermodynamic properties of section (2.1). Since the contribution  $\Omega_0/L^D$  depends only on  $g_0$  and is already known, we focus on the zero-temperature gaussian contribution  $\Omega_g^{(0)}/L^D$  to the grand potential. We calculate it substituting the finite-range spectrum (3.81) in the first term of the gaussian grand potential per unit of volume (2.5) and using  $D$ -dimensional spherical coordinates, namely

$$\frac{\Omega_g^{(0)}}{L^D} = \frac{S_D}{2(2\pi)^D} \int_0^{+\infty} dk k^{D-1} \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} \lambda(g_2, \mu/g_0) + 2\mu \right)} \quad (3.84)$$

In order to perform the dimensional regularization of this ultraviolet divergent integral, we use the adimensional integration variable  $t = \hbar^2 k^2 \lambda(g_2, \mu/g_0)/(4m\mu)$  and we extend the spatial dimension  $D$  of the system to the complex dimension  $\mathcal{D} = D - \varepsilon$ , where  $\varepsilon$  is infinitesimal. We obtain

$$\frac{\Omega_g^{(0)}}{L^D} = \frac{\kappa^\varepsilon \mu}{\Gamma[\mathcal{D}/2]} \frac{1}{\lambda(g_2, \mu/g_0)^{1/2}} \left( \frac{m\mu}{\pi \hbar^2 \lambda(g_2, \mu/g_0)} \right)^{\mathcal{D}/2} \int_0^{+\infty} dt t^{\frac{\mathcal{D}-1}{2}} (1+t)^{1/2} \quad (3.85)$$

By the means of the Euler beta and gamma functions and expanding the factors for small  $\varepsilon$ , we get the regularized zero-temperature gaussian grand potential  $\Omega_g^{(0)}/L^D$  for bosons with zero-range interaction

$$\begin{aligned} \frac{\Omega_g^{(0)}}{L^D} = & -\frac{\mu^{\frac{D+2}{2}}}{2\pi^{\frac{D+1}{2}}} \left( \frac{m}{\hbar^2} \right)^{D/2} \frac{1}{\lambda(g_2, \mu/g_0)^{\frac{D+1}{2}}} \left[ 1 + \frac{\varepsilon}{2} \ln \left( \frac{\pi \hbar^2 \kappa^2 \lambda(g_2, \mu/g_0)}{m\mu} \right) + o(\varepsilon^2) \right] \cdot \\ & \frac{\Gamma[(D - \varepsilon + 1)/2] \Gamma[(\varepsilon - D - 2)/2]}{\Gamma[(D - \varepsilon)/2]} \end{aligned} \quad (3.86)$$

that reproduces exactly the expression of Eq. (3.12) obtained in the zero-range interaction case if  $g_2 = 0$ .

Before substituting the spatial dimension  $D$  in  $\Omega_g^{(0)}/L^D$ , we calculate the gaussian thermal grand potential  $\Omega_g^{(T)}/L^D$ , which, according to the second term of Eq. (2.5), is given by

$$\frac{\Omega_g^{(T)}}{L^D} = \frac{S_D}{\beta(2\pi)^D} \int dk k^{D-1} \ln \left( 1 - e^{-\beta E_{k, \text{finite-range}}} \right) \quad (3.87)$$

where we have used  $D$ -dimensional spherical coordinates, since the integrand depends only on  $|\vec{k}|$ . Performing an integration by parts we rewrite it as

$$\frac{\Omega_g^{(T)}}{L^D} = -\frac{1}{2^{D-1} \pi^{D/2} \Gamma[D/2] D} \int_0^{+\infty} dk k^D \frac{\partial E_{k, \text{finite-range}}}{\partial k} \frac{1}{e^{\beta E_{k, \text{finite-range}}} - 1} \quad (3.88)$$

Since the analytical integration of  $\Omega_g^{(T)}/L^D$  is possible only in the low-temperature regime, we define the integration variable  $x = \beta E_{k,\text{finite-range}}$ , such as the thermal gaussian grand potential becomes

$$\frac{\Omega_g^{(T)}}{L^D} = -\frac{1}{2^{D-1}\pi^{D/2}\Gamma[D/2] \beta D} \int_0^{+\infty} dx k(x)^D \frac{1}{e^x - 1} \quad (3.89)$$

where we define  $k(x)$  as

$$k(x) = \sqrt{\frac{2m\mu}{\hbar^2\lambda(g_2, \mu/g_0)}} \sqrt{-1 + \sqrt{1 + \frac{(k_B T)^2 \lambda(g_2, \mu/g_0) x^2}{\mu^2}}} \quad (3.90)$$

Let us calculate explicitly the gaussian zero-temperature grand potential (3.86) and the gaussian thermal grand potential (3.89) in  $D = 3$  and in  $D = 2$ .

### D=3

In the three-dimensional case, it is sufficient to substitute the dimension  $D = 3$  and put  $\varepsilon = 0$  in Eq. (3.86), to get the regularized zero-temperature gaussian grand potential as

$$\frac{\Omega_g^{(0)}}{L^3} = \frac{8}{15\pi^2} \left(\frac{m}{\hbar^2}\right)^{3/2} \frac{\mu^{5/2}}{\lambda(g_2, \mu/g_0)^2} \quad (3.91)$$

The thermal gaussian grand potential is obtained as the integral of the low-temperature expansion of the integrand in (3.89), which in  $D = 3$  gives

$$\frac{\Omega_g^{(T)}}{L^3} = -\frac{\pi^2}{90} \left(\frac{m}{\hbar^2}\right)^{3/2} \frac{(k_B T)^4}{\mu^{3/2}} \left(1 - \frac{5\pi^2}{7} \frac{(k_B T)^2 \lambda(g_2, \mu/g_0)}{\mu^2}\right) + o((k_B T)^7) \quad (3.92)$$

Notice that the lowest order term in the temperature of  $\Omega_g^{(T)}/L^3$  does not depend on  $\lambda(g_2, \mu/g_0)$ , meaning that, in the low-temperature limit, the correction of the zero-range interaction with a finite-range term does not influence the thermal part of the gaussian grand potential.

The beyond mean field grand potential per unit of volume, up to gaussian corrections reads

$$\frac{\Omega}{L^3} = -\frac{\mu^2}{2g_0} + \frac{8}{15\pi^2} \left(\frac{m}{\hbar^2}\right)^{3/2} \frac{\mu^{5/2}}{\lambda(g_2, \mu/g_0)^2} - \frac{\pi^2}{90} \left(\frac{m}{\hbar^2}\right)^{3/2} \frac{(k_B T)^4}{\mu^{3/2}} \left(1 - \frac{5\pi^2}{7} \frac{(k_B T)^2 \lambda(g_2, \mu/g_0)}{\mu^2}\right) \quad (3.93)$$

which reproduces a previously known result [10].

**D=2**

Despite the dimensional regularization procedure, the substitution of  $D = 2$  in the zero-temperature grand potential (3.86) shows a  $o(\varepsilon^{-1})$  divergence of  $\Gamma[\varepsilon]$ . We treat the divergence expanding the grand potential for small  $\varepsilon$ , obtaining

$$\frac{\Omega_g^{(0)}}{L^2} = -\frac{m\mu^2}{2\pi^{3/2}\lambda(g_2, \mu/g_0)^{3/2}\hbar^2} \left[ \frac{\pi^{1/2}}{2} \frac{1}{\varepsilon} + \frac{\pi^{1/2}}{8} (\ln(16) - 2\gamma - 1) + \frac{\pi^{1/2}}{4} \ln \left( \frac{\pi\lambda(g_2, \mu/g_0)\hbar^2\kappa^2}{m\mu} \right) + o(\varepsilon) \right] \quad (3.94)$$

According to the prescriptions of section (1.4), we delete the  $o(\varepsilon^{-1})$  divergence in the square bracket and rearrange the other contributions in the form

$$\frac{\Omega_g^{(0)}}{L^2} = -\frac{m\mu^2}{8\pi\hbar^2\lambda(g_2, \mu/g_0)^{3/2}} \ln \left( \frac{4\pi\hbar^2\kappa^2}{m\mu e^{\gamma+1/2}} \lambda(g_2, \mu/g_0) \right) \quad (3.95)$$

Following the procedure introduced for bosons with zero-range interaction, we now write the sum of the order parameter grand potential (1.62) and the zero-temperature gaussian grand potential calculated above as

$$\frac{\Omega_0}{L^2} + \frac{\Omega_g^{(0)}}{L^2} = -\frac{\mu^2}{2g_0} - \frac{m\mu^2}{8\pi\hbar^2\lambda(g_2, \mu/g_0)^{3/2}} \ln \left( \frac{4\pi\hbar^2\kappa^2}{m\mu e^{\gamma+1/2}} \lambda(g_2, \mu/g_0) \right) \quad (3.96)$$

In order to reproduce the contact interaction result in the case of zero finite-range correction  $g_2 = 0$ , we identify the energy cutoff  $\epsilon_0$  as

$$\epsilon_0 = \frac{4\pi\hbar^2\kappa^2}{m \exp\left(\gamma - \frac{4\pi\hbar^2\lambda(g_2, \mu/g_0)^{3/2}}{mg_0}\right)} \quad (3.97)$$

to get the zero-temperature grand potential

$$\frac{\Omega_0}{L^2} + \frac{\Omega_g^{(0)}}{L^2} = -\frac{m\mu^2}{8\pi\hbar^2\lambda(g_2, \mu/g_0)^{3/2}} \left[ \ln \left( \frac{\epsilon_0}{\mu} \lambda(g_2, \mu/g_0) \right) - \frac{1}{2} \right] \quad (3.98)$$

Notice that this equation corrects a previous result [43], in which  $\lambda(g_2, \mu/g_0)$  inside the logarithm has been neglected in the renormalization procedure.

At last, we calculate the thermal gaussian contribution  $\Omega_g^{(T)}/L^2$ , expanding the integrand of the equation (3.89) in the zero-temperature limit and choosing  $D = 2$ , obtaining

$$\frac{\Omega_g^{(T)}}{L^2} = -\frac{m}{4\pi\hbar^2} (k_B T)^3 \left( \frac{2\zeta(3)}{\mu} - \frac{6\zeta(5)\lambda(g_2, \mu/g_0)}{\mu^3} (k_B T)^2 \right) + o((k_B T)^6) \quad (3.99)$$

The renormalized two-dimensional grand potential per unit of area in the low-temperature regime is obtained from Eqs. (3.98) and (3.99) as

$$\frac{\Omega}{L^2} = -\frac{m\mu^2}{8\pi\hbar^2\lambda(g_2, \mu/g_0)^{3/2}} \left[ \ln\left(\frac{\epsilon_0}{\mu}\lambda(g_2, \mu/g_0)\right) - \frac{1}{2} \right] - \frac{m}{4\pi\hbar^2}(k_B T)^3 \left( \frac{2\zeta(3)}{\mu} - \frac{6\zeta(5)\lambda(g_2, \mu/g_0)}{\mu^3}(k_B T)^2 \right) \quad (3.100)$$

### 3.2.2 Quantum depletion and thermal condensate fraction

Let us calculate the zero-temperature density  $n(n_0, T=0)$  of bosons with finite-range interaction implementing Eq. (2.21), which we remind to be

$$n(n_0, T=0) = n_0 + f_g^{(0)}(n_0) \quad (3.101)$$

In full analogy with the previous section, we need to calculate the zero-temperature correction  $f_g^{(0)}(n_0)$  to the number density, which, using  $D$ -dimensional spherical coordinates, is given by Eq. (2.24) in which the finite-range interaction (3.75) is substituted, namely

$$f_g^{(0)}(n_0) = \frac{S_D}{2(2\pi)^D} \int_0^{+\infty} dk k^{D-1} \frac{\frac{\hbar^2 k^2}{2m} \tilde{\lambda}(g_2, n_0) + g_0 n_0}{\sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} \lambda(g_2, n_0) + 2g_0 n_0 \right)}} \quad (3.102)$$

where we define the functions

$$\lambda(g_2, n_0) = 1 + \frac{4m}{\hbar^2} g_2 n_0 \quad \tilde{\lambda}(g_2, n_0) = 1 + \frac{2m}{\hbar^2} g_2 n_0 \quad (3.103)$$

which, if a pure contact interaction is restored by putting  $g_2 = 0$ , give  $\lambda(0, n_0) = \tilde{\lambda}(0, n_0) = 1$

To regularize the ultraviolet divergence of  $f_g^{(0)}(n_0)$  we obtain an adimensional integral using the integration variable  $t = \hbar^2 k^2 \lambda(g_2, n_0) / (4m g_0 n_0)$ , then we extend the spatial dimension  $D$  to the complex value  $\mathcal{D} = D - \varepsilon$  and we perform dimensional regularization. We find

$$f_g^{(0)}(n_0) = \frac{\kappa^\varepsilon \lambda(g_2, n_0)^{1/2}}{4\Gamma(\mathcal{D}/2)} \left( \frac{m g_0 n_0}{\pi \hbar^2 \lambda(g_2, n_0)} \right)^{\mathcal{D}/2} \left[ 2 \frac{\tilde{\lambda}(g_2, n_0)}{\lambda(g_2, n_0)} B\left(\frac{\mathcal{D}+1}{2}, \frac{-\mathcal{D}}{2}\right) + B\left(\frac{\mathcal{D}-1}{2}, \frac{2-\mathcal{D}}{2}\right) \right] \quad (3.104)$$

We explicitly substitute the Euler beta function expressed in terms of the gamma function (1.77) and we employ the recursive relation (1.76) for the gamma function. More-

over, we also substitute the explicit definition of  $\tilde{\lambda}(g_2, n_0)$ , to get the regularized zero-temperature number density correction

$$f_g^{(0)}(n_0) = \frac{1}{8\pi^{1/2}\lambda(g_2, n_0)^{\frac{D-1}{2}}} \left( \frac{mg_0n_0}{\pi\hbar^2} \right)^{D/2} \left( \frac{\pi\hbar^2\kappa^2}{mg_0n_0} \lambda(g_2, n_0) \right)^{\varepsilon/2} \cdot \left[ (D - \varepsilon - 2) - \frac{8m}{\hbar^2} \frac{g_2n_0}{\lambda(g_2, n_0)} \right] \frac{\Gamma[\frac{\varepsilon-D}{2}] \Gamma[\frac{D-\varepsilon-1}{2}]}{\Gamma[\frac{D-\varepsilon}{2}]} \quad (3.105)$$

Notice that if we put  $g_2 = 0$  the second term in the square bracket becomes zero, while the first reproduces the known contact-interaction correction (3.32).

Before substituting the dimension  $D$  in  $f_g^{(0)}(n_0)$ , let us also calculate the finite-temperature gaussian correction  $f_g^{(T)}(n_0)$  to the number density. Following its definition (2.27) we write, in the case of finite-range interaction

$$f_g^{(T)}(n_0) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{e^{\beta E_k(n_0)} - 1} \frac{\frac{\hbar^2 k^2}{2m} \lambda(g_2, n_0) + n_0 g_0}{E_k(n_0)} \quad (3.106)$$

where

$$E_k(n_0) = \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} \lambda(g_2, n_0) + 2n_0 g_0 \right)} \quad (3.107)$$

is the excitation spectrum. Using  $D$ -dimensional spherical coordinates and changing the integration variable with  $x = \beta E_k(n_0)$ , we obtain the finite-temperature number density contribution

$$f_g^{(T)}(n_0) = \frac{S_D}{(2\pi)^D k_B T} \int_0^{+\infty} dx \frac{dk(x)}{dx} \frac{k(x)^{D-1}}{x(e^x - 1)} \left( \frac{\hbar^2 k(x)^2}{2m} \lambda(g_2, n_0) + n_0 g_0 \right) \quad (3.108)$$

where  $k(x)$  is given by

$$k(x) = \sqrt{\frac{2mn_0g_0}{\hbar^2\lambda(g_2, n_0)}} \sqrt{-1 + \sqrt{1 + \frac{(k_B T)^2 \lambda(g_2, n_0) x^2}{n_0^2 g_0^2}}} \quad (3.109)$$

Let us now explicitly substitute in the equations (3.105) and (3.108) for  $f_g^{(0)}(n_0)$  and  $f_g^{(T)}(n_0)$  the dimensions  $D = 3$  and  $D = 2$ .

### D=3

The three-dimensional zero-temperature contribution  $f_g^{(0)}(n_0)$  does not show  $o(\varepsilon^{-1})$  divergences of the gamma functions, therefore we simply put  $D = 3$  and  $\varepsilon = 0$  in Eq. (3.105), obtaining

$$f_g^{(0)}(n_0) = \frac{1}{3\pi^2\lambda(g_2, n_0)} \left( \frac{mg_0n_0}{\hbar^2} \right)^{3/2} \left[ 1 - \frac{8m}{\hbar^2} \frac{g_2n_0}{\lambda(g_2, n_0)} \right] \quad (3.110)$$

Since we expect that the finite-range interaction strength  $g_2$  constitutes a small correction to the contact interaction, we expand the previous expression for small values of  $g_2$ , to find the zero-temperature density correction

$$f_g^{(0)}(n_0) = \frac{1}{3\pi^2} \left( \frac{mg_0 n_0}{\hbar^2} \right)^{3/2} \left[ 1 - \frac{12m}{\hbar^2} g_2 n_0 \right] \quad (3.111)$$

In this limit, the zero-temperature density  $n(n_0, T=0)$  of Eq. (2.21) is given by

$$n = n_0 + \frac{1}{3\pi^2} \left( \frac{mg_0 n_0}{\hbar^2} \right)^{3/2} \left[ 1 - \frac{12m}{\hbar^2} g_2 n_0 \right] \quad (3.112)$$

With the scattering theory, we relate the contact interaction strength  $g_0$  and the finite-range interaction strength  $g_2$  with the three-dimensional scattering length  $a_s$  and of the effective range of the interaction  $r_{\text{eff}}$  by [10]

$$g_0 = \frac{4\pi\hbar^2 a_s}{m} \quad g_2 = \frac{2\pi\hbar^2 a_s^2 r_{\text{eff}}}{m} \quad (3.113)$$

The substitution of these parameters in the density (3.112) leads to an extension of the three-dimensional result (3.39) obtained in the previous section, and gives

$$n = n_0 \left[ 1 + \frac{8}{3\sqrt{\pi}} (n_0 a_s^3)^{1/2} - 64\sqrt{\pi} \frac{r_{\text{eff}}}{a_s} (n_0 a_s^3)^{3/2} \right] \quad (3.114)$$

Notice that this result coincides with the one obtained by Andersen [4] unless than a factor 2 in the finite-range correction.

We now define the rescaled adimensional density  $\tilde{n}$  and the rescaled condensate density  $\tilde{n}_0$

$$\tilde{n} = n a_s^3 \quad \tilde{n}_0 = n_0 a_s^3 \quad (3.115)$$

such as the rescaled density  $\tilde{n}$  is given by

$$\tilde{n} = \tilde{n}_0 \left[ 1 + \frac{8}{3\sqrt{\pi}} (\tilde{n}_0)^{1/2} - 64\sqrt{\pi} \frac{r_{\text{eff}}}{a_s} (\tilde{n}_0)^{3/2} \right] \quad (3.116)$$

The zero-temperature condensate fraction  $\tilde{n}_0/\tilde{n} = n_0/n$  is obtained as the numerical solution of Eq. (3.116) and is reported in Fig. (3.7). In particular, we compare the zero-range interaction condensate fraction, as obtained from our Eq. (3.39) (blue solid line) and by Bogoliubov (3.40) (red dashed line), with the result for finite-range interaction of Eq. (3.116) (black dot-dashed line). In the plot, we have chosen  $r_{\text{eff}}/a_s = -10$ , thus the inclusion of a finite-range interaction correction in the three-dimensional uniform Bose gas is relevant in the specific regime  $r_{\text{eff}} < -a_s$ . Fig. (3.7) also shows that the finite-range interaction leads to relevant corrections for a gas parameter  $a_s \gtrsim 10^{-3}$ . In the next section we will show that, for two-dimensional bosons, the finite-range corrections are less pronounced.

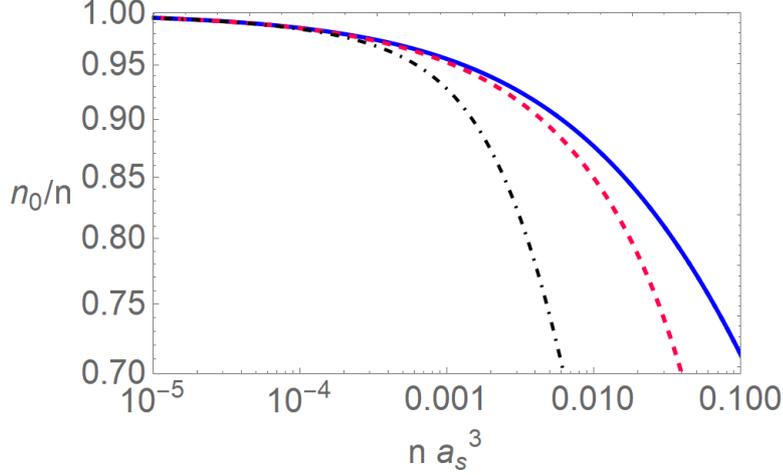


Figure 3.7: Zero-temperature condensate fraction  $n_0/n$  as a function of the gas parameter  $na_s^3$  in  $D = 3$ . Here we represent the finite-range condensate fraction (black dot-dashed line) for the effective range value  $r_{\text{eff}}/a_s = -10$  in comparison with our contact interaction result (blue solid line) and the Bogoliubov's result (red dashed line). With a weak dependence on the choice of  $r_{\text{eff}}$ , the finite-range correction becomes relevant for values of the gas parameter  $na_s^3$  greater than  $10^{-3}$ .

We now calculate in  $D = 3$  the convergent integral of the finite-temperature gaussian contribution  $f_g^{(T)}(n_0)$  in the low-temperature limit, namely

$$f_g^{(T)}(n_0) = \frac{(k_B T)^2}{12(n_0 g_0)^{1/2}} \left( \frac{m}{\hbar^2} \right)^{3/2} \left( 1 - \frac{\pi^2 (k_B T)^2 \lambda(g_2, n_0)}{20(n_0 g_0)^2} \right) \quad (3.117)$$

Notice that, differently from the zero-temperature density contribution  $f_g^{(0)}(n_0)$ , the finite-range correction is not crucial around  $T = 0$ , since it involves only higher powers of the temperature with respect to the leading phononic term.

Putting together the density contributions of Eq. (3.110), and (3.117) inside the number density (2.25) we find the low-temperature density  $n(n_0, T)$  for bosons with finite-range interaction in  $D = 3$

$$n(n_0, T) = n_0 + \frac{1}{3\pi^2 \lambda(g_2, n_0)} \left( \frac{m g_0 n_0}{\hbar^2} \right)^{3/2} \left[ 1 - \frac{8m}{\hbar^2} \frac{g_2 n_0}{\lambda(g_2, n_0)} \right] + \frac{(k_B T)^2}{12(n_0 g_0)^{1/2}} \left( \frac{m}{\hbar^2} \right)^{3/2} \quad (3.118)$$

We now use this equation to obtain the condensate fraction  $n_0/n$  in terms of the temperature, first in an approximated explicit form, then in the exact but implicit form.

Let us substitute the expression of  $\lambda(g_2, n_0)$  given by Eq. (3.82) in the number density (3.118). We also replace the three-dimensional interaction strengths  $g_0$  and  $g_2$  with their values (3.113) identified with scattering theory as functions of the s-wave scattering length  $a_s$  and the effective range  $r_{\text{eff}}$ . Dividing both sides of the previous equation for

the number density  $n$ , the last equation can be recast in terms of the thermal condensate fraction  $n_0/n$  as

$$1 = \left(\frac{n_0}{n}\right) + \frac{8}{3\sqrt{\pi}}(na_s^3)^{1/2}\left(\frac{n_0}{n}\right)^{3/2} \frac{1 - 8(na_s^3)r_{\text{eff}}/a_s(n_0/n)}{(1 + 8(na_s^3)r_{\text{eff}}/a_s(n_0/n))^2} + \frac{1}{24\sqrt{\pi}} \frac{(k_B T)^2}{(na_s^3)^{1/6}} \left(\frac{m}{\hbar^2 n^{2/3}}\right)^2 \left(\frac{n}{n_0}\right)^{1/2} \quad (3.119)$$

Under the hypothesis that the bosons are very weakly-interacting and working in the limit of very low temperatures, we can approximate  $(n_0/n)^{1/2} \approx 1$  in the third term and  $(n_0/n) \approx 1$  in the second term at the right-hand side. We get the condensate fraction expanding for low values of the gas parameter  $na_s^3$  and for vanishing effective range  $r_{\text{eff}} \rightarrow 0$ , namely

$$\frac{n_0}{n} = 1 - \frac{8}{3\sqrt{\pi}}(na_s^3)^{1/2} \left[ 1 - 24\pi(na_s^3)r_{\text{eff}}/a_s + \frac{1}{64} \frac{(k_B T)^2}{(na_s^3)^{2/3}} \left(\frac{m}{\hbar^2 n^{2/3}}\right)^2 \right] \quad (3.120)$$

A more general relation, however, can be obtained without making any approximation in the Eq. (3.119) with the simple definition of the adimensional rescaled temperature  $T^*$  as in Eq. (3.47), obtaining the implicit equation

$$\left(\frac{n_0}{n}\right) + \frac{8}{3\sqrt{\pi}}(na_s^3)^{1/2}\left(\frac{n_0}{n}\right)^{3/2} \frac{1 - 8(na_s^3)r_{\text{eff}}/a_s(n_0/n)}{(1 + 8(na_s^3)r_{\text{eff}}/a_s(n_0/n))^2} + \frac{1}{24\sqrt{\pi}} \frac{T^{*2}}{(na_s^3)^{1/6}} \left(\frac{n}{n_0}\right)^{1/2} - 1 = 0 \quad (3.121)$$

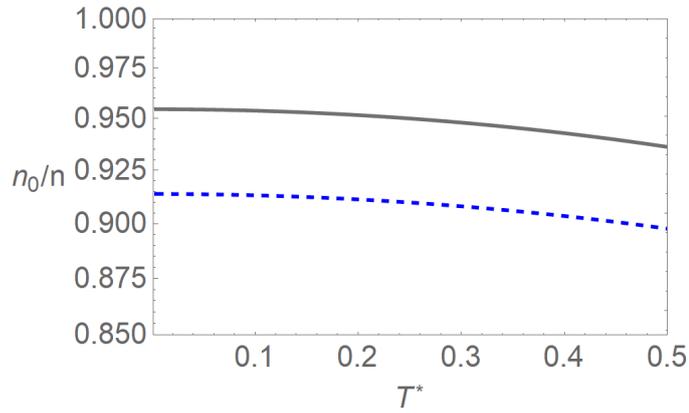


Figure 3.8: Condensate fraction  $n_0/n$  in  $D = 3$  as a function of the adimensional temperature  $T^* = k_B T/E_r$ , with  $E_r = \hbar^2 n/m$ . Here we compare the previous result for bosons with zero-range interaction in  $D = 3$  (upper curve) and the condensate fraction for bosons with finite-range interaction (lower curve), for which we choose  $r_{\text{eff}}/a_s = -10$ .

In Fig. (3.8) we plot the condensate fraction  $n_0/n$  as a function of the rescaled temperature  $T^*$ , comparing the contact-interaction case (upper curve) with the finite-range interaction case (lower curve) obtained from the numerical solution of Eq. (3.121). Despite giving different quantum depletion, the behavior for  $T > 0$  of the curves is the same, since, in the low-temperature expansion of  $f_g^{(T)}(n_0)$ , we have only considered the lowest order term in  $k_B T$  (which does not depend on the finite-range correction  $g_2$ ).

## D=2

In the two-dimensional case we obtain the regularized zero-temperature density contribution  $f_g^{(0)}(n_0)$  from Eq. (3.105) where  $D = 2$  is substituted. Notice that  $f_g^{(0)}(n_0)$  is the sum of the two terms inside the square bracket of (3.105): while the first term is finite due to the limit (3.50), the second term is divergent, but can be easily calculated expanding for small  $\varepsilon$  and deleting the divergent  $o(\varepsilon^{-1})$  terms. The regularized zero-temperature density contribution  $f_g^{(0)}(n_0)$  reads

$$f_g^{(0)}(n_0) = \frac{mg_0 n_0}{4\pi\hbar^2} \frac{1}{\lambda(g_2, n_0)^{3/2}} \left[ \tilde{\lambda}(g_2, n_0) + \frac{2m}{\hbar^2} g_2 n_0 \ln \left( \frac{2\varepsilon_0}{g_0 n_0} \lambda(g_2, n_0) \right) \right] \quad (3.122)$$

where we identify the ultraviolet energy scale  $\varepsilon_0$  as

$$\varepsilon_0 = \frac{4\pi\hbar^2 \kappa^2}{me^\gamma} \quad (3.123)$$

If the finite-range interaction strength  $g_2$ , as we suppose, constitutes a small correction of the zero-range term  $g_0$ , we expand the previous equation for small values of the adimensional parameter  $\frac{2m}{\hbar^2} g_2 n_0$

$$f_g^{(0)}(n_0) = \frac{1}{4\pi} \frac{mg_0 n_0}{\hbar^2} + \frac{1}{2\pi} \left( \frac{m}{\hbar^2} \right)^2 g_0 g_2 n_0^2 \left[ \ln \left( \frac{2\varepsilon_0}{g_0 n_0} \right) - 2 \right] \quad (3.124)$$

In this limit, the zero-temperature density  $n(n_0, T = 0)$  of Eq. (2.21) reads

$$n(n_0, T = 0) = n_0 + \frac{1}{4\pi} \frac{mg_0 n_0}{\hbar^2} + \frac{1}{2\pi} \left( \frac{m}{\hbar^2} \right)^2 g_0 g_2 n_0^2 \left[ \ln \left( \frac{2\varepsilon_0}{g_0 n_0} \right) - 2 \right] \quad (3.125)$$

To get an approximated - but explicit - formula for the condensate fraction  $n_0/n$  of a two-dimensional bosonic system with finite-range interaction, we consider the weakly-interacting regime in which  $g_0, g_2 \rightarrow 0$ . In this regime we can approximate  $n_0^2 \approx n_0 n$  in the third term of Eq. (3.125) and  $n_0 \approx n$  inside the logarithm. Moreover, we also substitute back the definition of  $\varepsilon_0$ , choosing the ultraviolet wavevector cutoff  $\kappa$  proportional to the inverse of the scattering length  $a_s$ , namely  $\kappa = 2\pi/a_s$ . Thus, we get the condensate fraction  $n_0/n$  from Eq. (3.125) as

$$\frac{n_0}{n} = \frac{1}{1 + \frac{1}{4\pi} \frac{mg_0}{\hbar^2} + \frac{1}{2\pi} \left( \frac{m}{\hbar^2} \right)^2 g_0 g_2 \ln \left( \frac{32\pi^3 \hbar^2}{mg_0 n a_s^2 e^{\gamma+2}} \right)} \quad (3.126)$$

Introducing the characteristic range  $R = 2\sqrt{|g_2/g_0|}$  of the interatomic potential, which can be explicitly calculated for different forms of interatomic potentials [43], we can use the scattering theory to identify the values of  $g_0$  and  $g_2$  as

$$g_0 = \frac{4\pi\hbar^2}{m} \frac{1}{|\ln(na_s^2)|} \quad g_2 = \frac{\pi\hbar^2}{m} \frac{R^2}{|\ln(na_s^2)|} \quad (3.127)$$

where  $g_0$  is given by Eq. (3.52) and  $g_2$  has been obtained through the definition of the characteristic range. We substitute these expressions in the Eq. (3.126) of the condensate fraction and, since we are limiting ourselves to the weakly-interacting regime, we expand the right-hand term as the geometric series. We get the approximated condensate fraction  $n_0/n$  as a function of the two-dimensional gas parameter  $na_s^2$  and the characteristic range  $R$

$$\frac{n_0}{n} = 1 - \frac{1}{|\ln(na_s^2)|} - \frac{2\pi(R/a_s)^2}{|\ln(na_s^2)|^2} \ln\left(\frac{8\pi^2|\ln(na_s^2)|}{na_s^2 e^{\gamma+2}}\right) \quad (3.128)$$

Notice that the zero-range interaction result by Schick of Eq. (3.54) is easily reproduced by setting  $R = 0$ .

We also derive a more general equation for  $n_0/n$ , even though in an implicit form, valid outside the very weakly-interacting regime. Substituting the values of  $g_0$  and  $g_2$  inside the number density (3.125) we get

$$n(n_0, T=0) = n_0 + \frac{n_0}{|\ln(na_s^2)|} + \frac{2\pi R^2 n_0^2}{|\ln(na_s^2)|^2} \ln\left(\frac{\epsilon_0 m |\ln(na_s^2)|}{2\pi\hbar^2 n_0 e^{2\gamma}}\right) \quad (3.129)$$

In order to plot  $n_0/n$  as a function of the gas parameter  $na_s^2$ , we employ the definitions of the rescaled density  $\tilde{n}$  and the rescaled condensate density  $\tilde{n}_0$  of Eq. (3.55) and we substitute back the value of  $\epsilon_0$  according its definition, choosing the ultraviolet wavevector cutoff value  $\kappa = 2\pi/a_s$ . We obtain the implicit equation for the zero-temperature condensate fraction

$$\frac{\tilde{n}_0}{\tilde{n}} + \frac{\tilde{n}_0}{\tilde{n}} \frac{1}{|\ln(\tilde{n})|} \left[ 1 + \frac{2\pi\tilde{n}_0 R^2 / a_s^2}{|\ln(\tilde{n})|} \ln\left(\frac{\kappa^2 a_s^2 (|\ln(\tilde{n})| + 4\pi\tilde{n}_0 R^2 / a_s^2)}{\tilde{n}_0 e^{\gamma+1}}\right) \right] - 1 = 0 \quad (3.130)$$

The zero-temperature condensate fraction  $\tilde{n}_0/\tilde{n} = n_0/n$  obtained from Eq. (3.130) is reported as the black dot-dashed line in Fig. (3.9), in comparison with the contact interaction result of Eq. (3.53) (blue solid line) and the result by Schick of Eq. (3.54) (red dashed line). In particular, we plot the finite-range curve for the effective range value of  $R = 2a_s$ .

We emphasize that an analytical formula for the condensate fraction of a bosonic system of cold atoms with a finite-range interaction in  $D = 2$  constitutes a new result, even if in the implicit form of Eq. (3.130).

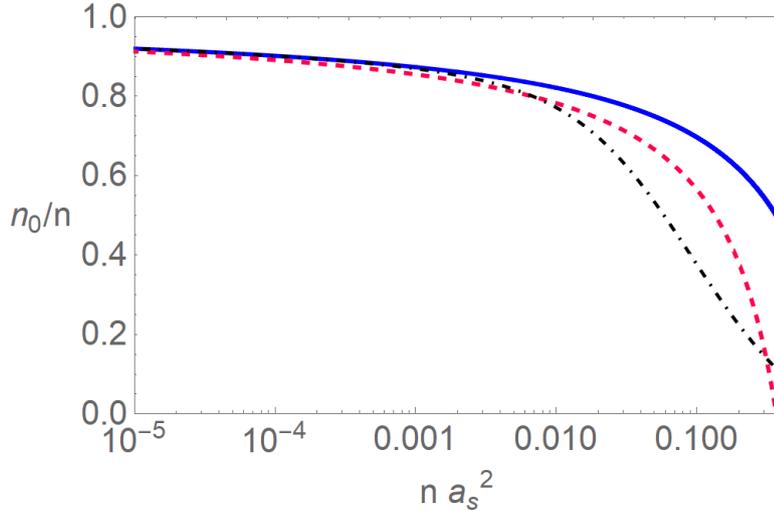


Figure 3.9: Condensate fraction  $n_0/n$  at  $T = 0$ , reported as a function of the gas parameter  $na_s^2$  in two-dimensional  $D = 2$  bosons. The red solid line is the condensate fraction for bosons with zero-range interaction, obtained from Eq. (3.56). The black dashed line is the condensate fraction for bosons with finite-range interaction given by Eq. (3.130) with a characteristic range  $R$  value given by  $R = 2a_s$  and a ultraviolet cutoff  $\kappa = 2\pi/a_s$ .

Finally, following the contact interaction case, we may want to calculate also the finite-temperature density contribution  $f_g^{(T)}(n_0)$ . However, substituting  $D = 2$  in Eq. (3.108) we find that

$$f_g^{(T)}(n_0) = \frac{mk_B T}{2\pi\hbar^2} \int_0^{+\infty} dx \frac{1}{e^x - 1} \quad (3.131)$$

is infrared divergent, as one could expect from the beginning. This result also shows that the inclusion of a finite-range interaction term is not relevant to the Mermin-Wagner theorem.

### 3.2.3 Superfluid density

Let us see how the superfluid density and the superfluid fraction of a bosonic system of cold atoms changes with respect to the zero-range interaction if we include a finite-range interaction contribution. Here we implement the superfluid density expression

$$n_s = n(n_0, T) - n_n(n_0, T) \quad (3.132)$$

Considering that  $n(n_0, T)$  is already known from the previous subsection, to derive the superfluid density  $n_s$  we only need to calculate the normal density  $n_n(n_0, T)$ , which, according to Eq. (2.76), reads

$$n_n(n_0, T) = \frac{\beta}{mD} \int \frac{d^D k}{(2\pi)^D} \hbar^2 k^2 \frac{e^{\beta E_k(n_0)}}{(e^{\beta E_k(n_0)} - 1)^2} \quad (3.133)$$

where  $E_k(n_0)$  is now the excitation spectrum defined in equation (3.107). Analogously to the finite-temperature density contribution  $f_g^{(T)}(n_0)$ , we use  $D$ -dimensional spherical coordinates and we set  $x = \beta E_k(n_0)$ , obtaining

$$n_n(n_0, T) = \frac{\beta \hbar^2 S_D}{mD(2\pi)^D} \int_0^{+\infty} dx \frac{dk}{dx}(x) k(x)^{D+1} \frac{e^x}{(e^x - 1)^2} \quad (3.134)$$

where  $k(x)$  is given by Eq. (3.109). The normal density can only be calculated analytically in the  $T \rightarrow 0$  limit, for which we need to explicitly consider the spatial dimensions  $D = 3$  and  $D = 2$ .

#### D=3

In the three-dimensional case we calculate  $n_n(n_0, T)$  expanding the integrand of Eq. (3.134) for low temperatures and performing the integration, in order to obtain

$$n_n(n_0, T) = \frac{2\pi^2}{45} \left( \frac{m}{\hbar^2} \right)^{3/2} \frac{(k_B T)^4}{(n_0 g_0)^{5/2}} \left( 1 - \frac{5\pi^2 \lambda(g_2, n_0) (k_B T)^2}{2(n_0 g_0)^2} + \frac{99\pi^4 \lambda(g_2, n_0)^2 (k_B T)^4}{8(n_0 g_0)^4} \right) + o(k_B T)^8 \quad (3.135)$$

Notice that, analogously to  $f_g^{(T)}(n_0)$  of Eq. (3.117), the finite-range correction term  $\lambda(g_2, n_0)$  interests only higher order terms with respect to the leading low-temperature contribution.

We now calculate the superfluid density  $n_s$  as a function of the condensate density  $n_0$  and the temperature  $T$  putting the number density  $n(n_0, T)$  and the normal density

$n_n(n_0, T)$  of Eqs. (3.118) and (3.135) inside the superfluid density (3.132), so that

$$n_s = n_0 + \frac{1}{3\pi^2\lambda(g_2, n_0)} \left(\frac{mg_0n_0}{\hbar^2}\right)^{3/2} \left[1 - \frac{8m}{\hbar^2} \frac{g_2n_0}{\lambda(g_2, n_0)}\right] + \frac{(k_B T)^2}{12(n_0g_0)^{1/2}} \left(\frac{m}{\hbar^2}\right)^{3/2} - \frac{(3\lambda(g_2, n_0) + 32)\pi^2}{720} \left(\frac{m}{\hbar^2}\right)^{3/2} \frac{(k_B T)^4}{(n_0g_0)^{5/2}} \quad (3.136)$$

Following the path outlined for the zero-range interaction, we now obtain an approximated expression of the superfluid fraction  $n_s/n$  considering the very weakly-interacting and low-temperature limit  $g_0, g_2 \rightarrow 0$ , in which we can approximate  $n_0 \approx n$ . First of all, we restore in the previous equation the number density  $n$  from its contributions and we include higher-order temperature contributions from the normal density (3.135). Then we explicitly substitute  $\lambda(g_2, n_0)$  from Eq. (3.103) and we employ the form of the interaction strengths  $g_0$  and  $g_2$  of Eq. (3.38), identified with the scattering theory in terms of  $a_s$  and  $r_{\text{eff}}$ . We obtain  $n_s/n$  as

$$\frac{n_s}{n} = 1 - \frac{1}{720\sqrt{\pi}} \left(\frac{m}{\hbar^2 n^{2/3}}\right)^4 \frac{(k_B T)^4}{(na_s^3)^{5/6}} \left[1 - \frac{5}{32} \left(\frac{m}{\hbar^2 n^{2/3}}\right)^2 \frac{(k_B T)^2}{(na_s^3)^{2/3}} \left(1 + 8\pi \frac{r_{\text{eff}}}{a_s} (na_s^3)\right)\right] \quad (3.137)$$

where the finite-range correction appears only in the next-to-leading term in the temperature.

Working outside the very weakly-interacting limit, a more general expression can be obtained performing the same steps described above, except for the strong approximation  $n_0 \approx n$ , and defining the adimensional rescaled temperature  $T^*$  as in Eq. (3.47). We find that the superfluid fraction  $n_s/n$  follows the implicit equation

$$\frac{n_s}{n} + \frac{T^{*4}}{720\sqrt{\pi}(na_s^3)^{5/6}} \left(\frac{n}{n_0}\right)^{5/2} \left[1 - \frac{5}{32} \frac{T^{*2}}{(na_s^3)^{2/3}} \left(\frac{n}{n_0}\right)^2 \left(1 + 8\pi \frac{r_{\text{eff}}}{a_s} \frac{n_0}{n} (na_s^3)\right)\right] - 1 = 0 \quad (3.138)$$

This equation can be solved numerically for a fixed value of the gas parameter  $na_s^3$  to obtain the three-dimensional superfluid fraction  $n_s/n$  of bosons with finite-range interaction as a function of the rescaled temperature  $T^*$ . In particular, the numerical solution of this equation requires the knowledge of the condensate fraction  $n_0/n$  in terms of the adimensional temperature  $T^*$ , which is given by Eq. (3.121) for the thermal condensate fraction derived in the previous subsection. In Fig. (3.10) we plot  $n_s/n$  in terms of  $T^*$  for various choices of the three-dimensional gas parameter  $na_s^3$ .

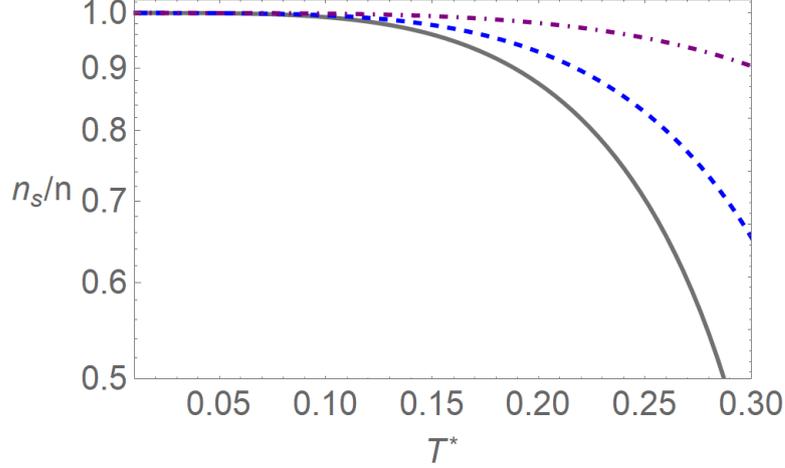


Figure 3.10: Superfluid fraction  $n_s/n$  of bosons with finite-range interaction in  $D = 3$  as a function of the rescaled temperature  $T^* = k_B T/E_r$ , where  $E_r = \hbar^2 n^{2/3}/m$  is an arbitrary energy scale. In particular, we represent  $n_s/n$  obtained from Eq. (3.138) for the explicit choice of three values of the gas parameter  $n a_s^3$ :  $10^{-6}$  (grey solid line),  $2 \cdot 10^{-6}$  (blue dashed line),  $10^{-5}$  (purple dot-dashed line) and for an effective range value  $r_{\text{eff}} = a_s$ .

## D=2

The two-dimensional normal density  $n_n(n_0, T=0)$  of bosons with finite-range interaction is obtained from the integration of Eq. (3.134), in which we expand the integrand in the low temperature limit, namely

$$n_n(n_0, T=0) = \frac{3m}{2\pi\hbar^2} \frac{(k_B T)^3}{(n_0 g_0)^2} \left( \zeta(3) - \frac{15\zeta(5)\lambda(g_2, n_0)}{(n_0 g_0)^2} (k_B T)^2 \right) + o(k_B T)^6 \quad (3.139)$$

Since the thermal contribution to the density  $f_g^{(T)}(n_0)$  is divergent in  $D = 2$ , we cannot express the superfluid density as a function of the condensate density  $n_0$  at a finite temperature  $T$ . Therefore, following the prescriptions of Eq. (2.85), we calculate the low-temperature superfluid density  $n_s$  as

$$n_s = n - \frac{3m}{2\pi\hbar^2} \frac{(k_B T)^3}{(g_0 n_0(n, T=0))^2} \left( \zeta(3) - \frac{15\zeta(5)\lambda(g_2, n_0(n, T=0))}{(g_0 n_0(n, T=0))^2} (k_B T)^2 \right) \quad (3.140)$$

where  $n_0(n, T=0)$  is the zero-temperature condensate density expressed in terms of the zero-temperature number density  $n(n_0, T=0)$  of Eq. (3.125), which we rewrite here in the equivalent form

$$n = n_0(n, T=0) + \frac{1}{4\pi} \frac{m g_0}{\hbar^2} n_0(n, T=0) + \frac{1}{2\pi} \left( \frac{m}{\hbar^2} \right)^2 g_0 g_2 n_0(n, T=0)^2 \left[ \ln \left( \frac{2\epsilon_0}{g_0 n_0(n, T=0)} \right) - 2 \right] \quad (3.141)$$

Following the path previously outlined, we calculate first an approximated expression of the superfluid fraction  $n_s/n$  in the very weakly interacting regime where  $g_0, g_2 \rightarrow 0$ , in which we can approximate the condensate density inside Eq. (3.140) as  $n_0(n, T=0) \approx n$ . In the context of this approximation, we also substitute in Eq. (3.140) the explicit form of the parameter  $\lambda(g_2, n_0)$  of Eq. (3.103). Moreover, we remember that the two-dimensional interaction strengths  $g_0$  and  $g_2$  are given by Eq. (3.127), obtained in terms of the s-wave scattering length  $a_s$  and the characteristic range  $R$  of the interatomic potential with the scattering theory. The approximated superfluid fraction  $n_s/n$  is given by

$$\frac{n_s}{n} = 1 - \frac{3}{32\pi^3} \left( \frac{m}{\hbar^2 n} \right)^3 |\ln(na_s^2)|^2 (k_B T)^3 \left[ \zeta(3) - \frac{15\zeta(5)}{16\pi^2} \left( \frac{m}{\hbar^2 n} \right)^2 \left( 1 + 4\pi \frac{R^2}{a_s^2} \frac{na_s^2}{|\ln(na_s^2)|} \right) |\ln(na_s^2)|^2 (k_B T)^2 \right] \quad (3.142)$$

Notice that, at the lowest order in the temperature, we reobtain the result of Eq. (3.70) which does not include any finite-range correction though.

The complete expression for the superfluid fraction  $n_s/n$  is calculated as a function of the adimensional temperature  $T^* = k_B T/E_r$ , with  $E_r = \hbar^2 n/m$ . We obtain it from Eq. (3.140) following the same steps described for the calculation of the approximated formula, except for the strong assumption  $n_0(n, T=0) \approx n$ , which is valid only in the very weakly-interacting regime where the quantum depletion is negligible and at low temperature, where the thermal depletion is not relevant. The exact but implicit equation for the superfluid fraction  $n_s/n$  in the low-temperature regime is given by

$$\frac{n_s}{n} + \frac{3T^{*3}}{32\pi^3} |\ln(na_s^2)|^2 \left( \frac{n}{n_0} \right)^2 \left[ \zeta(3) - \frac{15\zeta(5)T^{*2}}{16\pi^2} |\ln(na_s^2)|^2 \left( \frac{n}{n_0} \right)^2 \left( 1 + 4\pi \frac{R^2}{a_s^2} \frac{n_0}{n} \frac{na_s^2}{|\ln(na_s^2)|} \right) \right] - 1 = 0 \quad (3.143)$$

which, at the lowest order in the temperature expansion and for the characteristic range value  $R=0$ , reproduces the zero-range result of Eq. (3.72).

In Fig. (3.11) we plot the superfluid fraction  $n_s/n$  in terms of the adimensional temperature  $T^*$ , obtained from the numerical solution of Eq. (3.143). The plot is made considering different values of the two-dimensional gas parameter  $na_s^2$  and fixing the value of the characteristic range  $R$  of the interatomic potential to the value  $R = a_s$ .

As for the three-dimensional result of Eq. (3.138), we notice that the implicit equation for  $n_s/n$  depends on the condensate fraction  $n_0/n$ . Since the phenomenon of Bose-Einstein condensation cannot occur at finite temperature in  $D=2$ , there cannot be a nonzero finite-temperature result for  $n_0/n$ . Therefore, for the numerical solution of this equation, we employ the zero-temperature condensate fraction value, obtained from  $(n_0, T=0)$  of Eq. (3.141). We expect that this approximation is reliable, at least in the low-temperature regime where we suppose that our theory is correct.

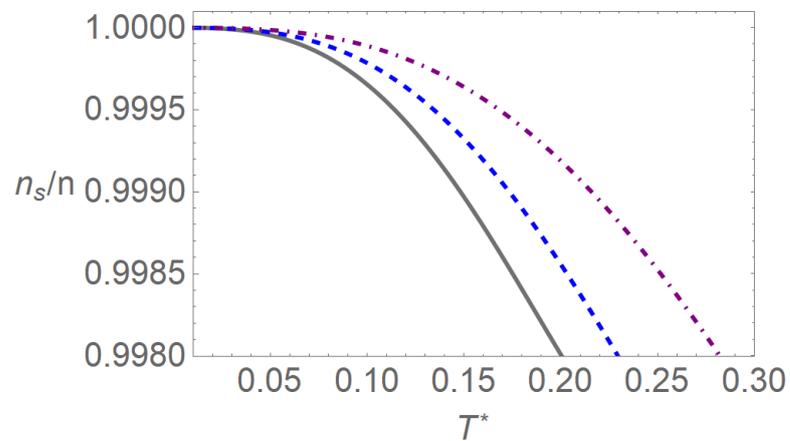


Figure 3.11: Superfluid fraction  $n_s/n$  of bosons with finite-range interaction in  $D = 2$  as a function of the rescaled temperature  $T^* = k_B T/E_r$ , where  $E_r = \hbar^2 n/m$  is an arbitrary energy scale. In particular, we represent  $n_s/n$  obtained from Eq. (3.138) for three values of the gas parameter  $na_s^3$ :  $10^{-4}$  (grey solid line),  $10^{-3}$  (blue dashed line),  $10^{-2}$  (purple dot-dashed line) and for an effective range value  $R = a_s$ .

## Chapter 4

# Conclusions and Outlook

In the thesis we have calculated the gaussian grand canonical partition function of a bosonic cold atoms system within the functional integration approach. In a consistent framework, we have been able to calculate the system number density and the superfluid density, expressed as functions of the condensate density and the temperature. We stress that, with an unified formalism and using the same procedure of dimensional regularization of divergent integrals, we have obtained these equations in  $D$  spatial dimensions and valid for a generic interaction potential  $V(\vec{k})$ .

This work reproduces important known results for uniform bosonic systems with a zero-range interaction. Among them we remind the condensate fraction of three-dimensional bosons as obtained by Bogoliubov in 1947 and the condensate fraction of two-dimensional bosons, according to the result by Schick of 1971. Also the three- and two- dimensional superfluid fraction is obtained coherently with the result of Landau.

Original contributions are provided in the case of bosons interacting with the finite-range interaction. First of all we include all the finite-range dependent terms in the gaussian grand potential in two dimensions, extending a previous result by Salasnich. Moreover, we have obtained, for the first time with the finite-range interaction, the condensate fraction in the two-dimensional case and the superfluid fraction in three and in two spatial dimensions. These results are derived as a low-temperature series expansion around  $T = 0$ , but an extension outside this specific limit can be provided with a numerical solution of the temperature-dependent integrals encountered, like the thermal number density contribution  $f_g^{(T)}(n_0)$  and the normal density  $n_n(n_0, T)$ .

At last, we discuss the choice of focusing on the description of three- and two-dimensional systems. While the study of three-dimensional interacting bosons is fundamental to understand the basic physical behavior of a bosonic cold atom system, the two-dimensional case has been introduced in the perspective of extending the thesis work. In fact, all the temperature-dependent results obtained here are valid in the low-temperature regime, however, if an extension to finite temperatures would be feasible, in two-dimensional

systems it is expected that the Berezinski-Kosterlitz-Thouless (BKT) topological phase transition occur. It has been shown that [39], at a finite temperature, the superfluid density jumps from a finite value to zero due to the unbinding of the vortex-antivortex couples, which destroy the quasi-long-range order of the system. The BKT phenomenology is indeed related to the vortical configurations of the phase of the complex field  $\psi(\vec{r}, \tau)$ , which constitute topological defects of the phase. These terms are not been explicitly included in the initial choice of the field parametrization, therefore we expect that the BKT phenomenology cannot be reproduced with this approach. For some suggestions on how to describe the BKT transition in our functional integration framework, we refer the reader to the Appendix B.

## Appendix A

# Evaluation of Matsubara summations

We illustrate here the procedure to calculate the summations over the bosonic Matsubara frequencies  $\omega_n$ , which are defined as

$$\omega_n = \frac{2\pi n}{\beta\hbar} \quad (\text{A.1})$$

where  $n \in \mathbb{Z}$  are integer numbers. The most common summation we need to perform is in the form

$$I[\xi_{\vec{k}}] = \frac{1}{2\beta} \sum_{n=-\infty}^{+\infty} \ln[\beta^2(\hbar^2\omega_n^2 + \xi_{\vec{k}}^2)] \quad (\text{A.2})$$

Using the properties of the logarithm and considering that the summation involves all  $n \in \mathbb{Z}$  integers, both positive and negative,  $I[\xi_{\vec{k}}]$  can also be rewritten in the useful form

$$I[\xi_{\vec{k}}] = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \ln[\beta(-i\hbar\omega_n + \xi_{\vec{k}})] \quad (\text{A.3})$$

Taking the derivative of  $I[\xi_{\vec{k}}]$  with respect to  $\xi_{\vec{k}}$  in the Eq. (A.2) we get

$$\frac{\partial I[\xi_{\vec{k}}]}{\partial \xi_{\vec{k}}} = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \frac{\xi_{\vec{k}}}{\hbar^2\omega_n^2 + \xi_{\vec{k}}^2} \quad (\text{A.4})$$

In the limit of zero temperature the difference

$$\Delta\omega = \omega_n - \omega_{n-1} = \frac{2\pi}{\beta\hbar} \xrightarrow{\beta \gg 1} d\omega \quad (\text{A.5})$$

becomes infinitesimal and we can substitute the sum over  $n$  with an integral over  $\omega$ , obtaining

$$\frac{\partial I[\xi_{\vec{k}}]}{\partial \xi_{\vec{k}}} = \frac{1}{\beta} \int_{-\infty}^{+\infty} d\omega \frac{\beta\hbar}{2\pi} \frac{\xi_{\vec{k}}}{\hbar^2\omega^2 + \xi_{\vec{k}}^2} = \frac{1}{2} \quad (\text{A.6})$$

which is the zero-temperature contribution to  $I[\xi_{\vec{k}}]$ . If the temperature is relatively low, but non-zero, we cannot substitute the sum in Eq. (A.4) with an integral, but we can rewrite it as

$$\frac{\partial I[\xi_{\vec{k}}]}{\partial \xi_{\vec{k}}} = \frac{\beta \xi_{\vec{k}}}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \frac{1}{n^2 + \left(\frac{\beta \xi_{\vec{k}}}{2\pi}\right)^2} \quad (\text{A.7})$$

and using the identity

$$\sum_{n=0}^{+\infty} \frac{1}{n^2 + a^2} = \frac{1 + \pi a \coth(\pi a)}{2a^2} \quad (\text{A.8})$$

we obtain

$$\frac{\partial I[\xi_{\vec{k}}]}{\partial \xi_{\vec{k}}} = \frac{1}{2} \coth\left(\frac{\beta \xi_{\vec{k}}}{2}\right) = \frac{1}{2} + \frac{1}{e^{\beta \xi_{\vec{k}}} - 1} \quad (\text{A.9})$$

We integrate this equation on  $\xi_{\vec{k}}$  and, setting the arbitrary constant resulting from the indefinite integral to zero (it is not dependent on physical parameters), we finally obtain the result of the summation over the Matsubara frequencies

$$I[\xi_{\vec{k}}] = \frac{\xi_{\vec{k}}}{2} + \frac{1}{\beta} \ln\left(1 - e^{-\beta \xi_{\vec{k}}}\right) \quad (\text{A.10})$$

## Appendix B

# Phase-amplitude field parametrization

The most simple way to describe a bosonic cold atoms system from an hydrodynamic point of view is constituted by the following parametrization of the bosonic complex field  $\psi(\vec{r}, \tau)$

$$\psi(\vec{r}, \tau) = \sqrt{\rho_0 + \delta\rho(\vec{r}, \tau)} e^{-i\theta(\vec{r}, \tau)} \quad (\text{B.1})$$

where  $\rho_0$  is the uniform density of the system in the broken symmetry phase,  $\delta\rho(\vec{r}, \tau)$  is the real field describing the density fluctuations and  $\theta(\vec{r}, \tau)$  is the real field of the Goldstone mode, describing massless phase fluctuations of the system.

Before substituting the field parametrization inside the lagrangian (1.36), let us analyze the fluctuation phase field  $\theta(\vec{r}, \tau)$ , which is defined in the compact interval  $[0, 2\pi]$  and is periodic of  $2\pi$ . This crucial fact implies that the spatial and the imaginary time derivatives of this field are not well-defined between  $2\pi$  and  $0$  which, from a mathematical point of view, constitutes the origin of many topological phenomena in condensed-matter physics. In two-dimensional systems the singularities of the phase field - the vortices - are responsible for the Berezinski-Kosterlitz-Thouless (BKT) transition, as explained by Kosterlitz and Thouless in 1973 [25].

Since the study of the BKT topological phase transition is beyond the scope of this thesis, we suppose to work in the very low-temperature regime, in which, the vortex-antivortex phenomenology does not play a fundamental role. At very low temperatures, due to the quasi-long-range phase coherence, the phase of the complex field  $\psi(\vec{r}, \tau)$  varies very slowly in space. Therefore, in the following we will assume that the domain of definition of the phase field  $\theta(\vec{r}, \tau)$  can be extended to the interval  $] - \infty, \infty[$  and that its spatial and time derivatives are well defined everywhere.

Taking into account the de-compactification of the phase field  $\theta(\vec{r}, \tau)$ , we substitute the

parametrization (B.1) in the lagrangian of Eq. (1.36), obtaining

$$\begin{aligned} \mathcal{L} = & -\mu\rho_0 - \mu\delta\rho + i\hbar\rho_0\partial_\tau\theta + \frac{\hbar^2}{8m\rho_0}(\nabla\delta\rho)^2 + \frac{\hbar^2\rho_0}{2m}(\nabla\theta)^2 + \\ & \frac{1}{2}\int d^D r' V(\vec{r}-\vec{r}')(\rho_0^2 + \delta\rho(\vec{r},\tau) + \delta\rho(\vec{r}',\tau) + \rho(\vec{r},\tau)\delta\rho(\vec{r}',\tau)) \end{aligned} \quad (\text{B.2})$$

Notice that some terms are omitted because, thinking the lagrangian inside the action (1.35), some terms can be integrated in the imaginary time  $\tau$  variable and give zero for the  $\tau$  periodicity of the fields. Moreover, we keep only terms up to second order in the fluctuation fields  $\delta\rho(\vec{r},\tau)$  and  $\theta(\vec{r},\tau)$ , namely we make a gaussian (one-loop) approximation.

Considering the lagrangian (B.2) inside the action  $S$  of Eq. (1.35), it is particularly convenient to express  $S$  in terms of the Fourier series of the fluctuation fields, namely

$$\begin{aligned} \delta\rho(\vec{r},\tau) &= \frac{1}{\sqrt{L^D}} \sum_{\vec{k}\omega_n} e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_n\tau} \delta\rho(\vec{k},\omega_n) \\ \theta(\vec{r},\tau) &= \frac{1}{\sqrt{L^D}} \sum_{\vec{k}\omega_n} e^{i\vec{k}\cdot\vec{r}} e^{-i\omega_n\tau} \theta(\vec{k},\omega_n) \\ \delta\rho(\vec{k},\omega_n) &= \frac{1}{\beta\hbar\sqrt{L^D}} \int_0^{\beta\hbar} d\tau \int_{L^D} d^D r e^{-i\vec{k}\cdot\vec{r}} e^{i\omega_n\tau} \delta\rho(\vec{r},\tau) \\ \theta(\vec{k},\omega_n) &= \frac{1}{\beta\hbar\sqrt{L^D}} \int_0^{\beta\hbar} d\tau \int_{L^D} d^D r e^{-i\vec{k}\cdot\vec{r}} e^{i\omega_n\tau} \theta(\vec{r},\tau) \end{aligned} \quad (\text{B.3})$$

Notice that, since we are supposing that the phase field  $\theta(\vec{r},\tau)$  is defined on  $\mathbb{R}$ , its Fourier components are non-numerable and can assume real values, therefore they can be treated like ordinary functional integral variables.

The action in the Fourier space is obtained by simply substituting these Fourier components in  $S$  and remembering the definition of the  $D+1$ -dimensional delta function (1.49) and the Fourier series (1.40) of the real space interaction potential. After some calculations, we get

$$S = S_0 + S_g \quad (\text{B.4})$$

where the homogeneous system action  $S_0$  is given by

$$S_0 = \beta\hbar L^D \left( -\mu n_0 + \frac{1}{2} g_0 n_0^2 \right) \quad (\text{B.5})$$

and the gaussian action  $S_g$  is calculated as

$$S_g = \beta\hbar \sum_{\vec{k}\omega_n} \left[ \frac{\hbar^2 k^2 \rho_0}{2m} \theta(k)\theta(-k) + \left( \frac{\hbar^2 k^2}{8m\rho_0} + \frac{V(\vec{k})}{2} \right) \delta\rho(k)\delta\rho(-k) + \hbar\omega_n \theta(k)\delta\rho(-k) \right] \quad (\text{B.6})$$

Let us focus on the functional integration of the action  $S_g$ , which, being quadratic in the fluctuation fields  $\delta\rho(k)$  and  $\theta(k)$ , can be rewritten in the matricial form

$$S_g = \frac{\hbar}{2} \sum_{\vec{k}\omega_n} (\theta(k) \quad \theta(-k) \quad \delta\rho(k) \quad \delta\rho(-k)) \mathbf{M}(k) \begin{pmatrix} \theta(k) \\ \theta(-k) \\ \delta\rho(k) \\ \delta\rho(-k) \end{pmatrix} \quad (\text{B.7})$$

with the  $4 \times 4$  matrix  $\mathbf{M}(k)$  given by

$$\mathbf{M}(k) = \beta \begin{pmatrix} 0 & \frac{\hbar^2 k^2 \rho_0}{m} & 0 & \hbar\omega_n \\ \frac{\hbar^2 k^2 \rho_0}{m} & 0 & -\hbar\omega_n & 0 \\ 0 & -\hbar\omega_n & 0 & \frac{\hbar^2 k^2}{4m\rho_0} + V(\vec{k}) \\ \hbar\omega_n & 0 & \frac{\hbar^2 k^2}{4m\rho_0} + V(\vec{k}) & 0 \end{pmatrix} \quad (\text{B.8})$$

Performing the functional integration of the real fluctuation fields  $\theta(k)$  and  $\delta\rho(k)$  we obtain the grand canonical partition function  $\mathcal{Z}$ , which, following the first chapter, can be written as

$$\mathcal{Z} = \mathcal{Z}_0 + \mathcal{Z}_g \quad (\text{B.9})$$

with  $\mathcal{Z}_0 = e^{-S_0/\hbar}$  and  $\mathcal{Z}_g$  given by

$$\mathcal{Z}_g = \prod_{\substack{\vec{k}\omega_n \\ k_z > 0}} [\det \mathbf{M}(k)]^{-1/2} \quad (\text{B.10})$$

In full analogy with the first chapter we employ the grand canonical partition function  $\mathcal{Z}$  to calculate the grand potential  $\Omega$  according to its definition (1.37), then, considering that the sum over Matsubara frequencies can be performed with the same prescriptions previously described (see Appendix A), we express the grand potential as the sum of three contributions

$$\Omega = \Omega_0 + \Omega_g^{(0)} + \Omega_g^{(T)} \quad (\text{B.11})$$

with the order parameter grand potential  $\Omega_0$  given by

$$\Omega_0 = \left( -\mu\rho_0 + \frac{1}{2}g_0\rho_0^2 \right) L^D \quad (\text{B.12})$$

while the zero- and the finite-temperature gaussian grand potential have the same exact form of the first chapter, namely they are given respectively by (1.67) and (1.68), but with the general excitation spectrum  $E_{\vec{k}}$  in the form

$$E_{\vec{k}} = \sqrt{\frac{\hbar^2 k^2}{2m} \left( \frac{\hbar^2 k^2}{2m} + 2\rho_0 V(\vec{k}) \right)} \quad (\text{B.13})$$

Notice that the saddle point condition  $\partial\Omega_0(\mu, \rho_0)/\partial\rho_0 = 0$  here implies

$$\rho_0 = \frac{\mu}{g_0} \quad (\text{B.14})$$

which, substituted in the excitation spectrum gives the Bogoliubov spectrum of Eq. (3.5) for bosons with zero-range interaction and the finite-range excitation spectrum of Eq. (3.81) for bosons with finite-range interaction.

The bosonic field parametrization (B.1) in terms of the real fields  $\theta(\vec{r}, \tau)$  and  $\delta\rho(\vec{r}, \tau)$  gives the same thermodynamic properties of the parametrization of the first chapter (1.43) which uses the complex field  $\eta(\vec{r}, \tau)$ . However, the description of the BKT phenomenology can be achieved in a simpler way, including in (B.1) a vortical phase field  $\theta_v(\vec{r}, \tau)$  which takes into account the topological defects of the phase field  $\theta(\vec{r}, \tau)$ , namely

$$\psi(\vec{r}, \tau) = \sqrt{\rho_0 + \delta\rho(\vec{r}, \tau)} e^{i(\theta(\vec{r}, \tau) + \theta_v(\vec{r}, \tau))} \quad (\text{B.15})$$

We expect that this parametrization can be used to correctly characterize the superfluid density of two-dimensional systems outside the low-temperature regime.

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